The classification of diagrams in perturbation theory

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Abstract

The derivation of scattering equations connecting the amplitudes obtained from diagrammatic expansions is of interest in many branches of physics. One method for deriving such equations is the classification-of-diagrams technique of Taylor. However, as we shall explain in this paper, there are certain points of Taylor's method which require clarification. Firstly, it is not clear whether Taylor's original method is equivalent to the simpler classification-of-diagrams scheme used by Thomas, Rinat, Afnan and Blankleider (TRAB). Secondly, when the Taylor method is applied to certain problems in a time-dependent perturbation theory it leads to the over-counting of some diagrams. This paper first restates Taylor's method, in the process uncovering reasons why certain diagrams might be double-counted in the Taylor method. It then explores how far Taylor's method is equivalent to the simpler TRAB method. Finally, it examines precisely why the double-counting occurs in Taylor's method, and derives corrections which compensate for this double-counting.

I. INTRODUCTION

The method of classification of diagrams, developed by Taylor, is a powerful technique by which equations connecting the amplitudes obtained from a field theory may be derived, without it being necessary to explicitly specify the Lagrangian of the theory in the derivation [1,2]. This model-independence has made the technique particularly useful in theories of mesons and baryons, where it is not practical to use the QCD Lagrangian and the best equivalent Lagrangian containing meson and baryon degrees of freedom is not yet known. Examples of the application of the technique to simple systems of nucleons and pions include the original work on equations for the $\pi\pi\pi$, $\pi\pi N$, πNN and NNN systems by Taylor himself [3]; the derivation of the $\pi N - \pi\pi N$ equations by Afnan and Pearce [4,5]; studies of pion photoproduction on both a single nucleon and the deuteron [6,7,8] and the derivation by Avishai and Mizutani [9,10], on the one hand, and Thomas, Rinat, Afnan and Blankleider on the other [11,12,13], of the $NN - \pi NN$ equations. This work on the $NN - \pi NN$ system raised at least two questions about the Taylor technique, both of which, despite the technique's widespread application, remain unanswered. Although these problems originally arose in the context of the $NN - \pi NN$ equations it should be clear that the questions themselves are quite general ones about the Taylor method and, as such, are relevant independent of the particular system and Lagrangian under consideration.

The first question arose because Taylor's original technique was somewhat modified and simplified by, first, Thomas and Rinat [12] and, second, Afnan and Blankleider [13,14], in order to make it more useful for time-ordered perturbation theory calculations. However, the equations obtained by Afnan and Blankleider for the $NN - \pi NN$ system [13] were exactly the same as those derived by Avishai and Mizutani[1], who used Taylor's original technique and a *time-dependent* perturbation theory [10]. Is this pure coincidence, or did Thomas and Rinat and Afnan and Blankleider (TRAB) discover a simplification of Taylor's technique? This question was posed and left unanswered by Avishai and Mizutani [10]. In this paper we answer it by explaining what assumptions must be made if the TRAB method is to produce the same equations as Taylor's original technique.

The second problem is that Taylor's method can lead to the double-counting of certain diagrams when it is applied in a time-dependent perturbation theory, such as covariant perturbation theory. This problem was first pointed out by Kowalski, Siciliano and Thaler who showed that there was double-counting in some models of pion absorption on nuclei [15]. While Kowalski et al. did not refer specifically to Taylor's method, the double-counting problem certainly arises when one applies the classification-of-diagrams technique to the problem of summing all possible diagrams contributing to, say, pion absorption on the deuteron. If one applies the Taylor method, as described below, to pion absorption on the deuteron, one obtains contributions from both of the diagrams in Fig. 1. In this figure $t^{(1)}$ is the πN t-matrix with the s-channel pole part removed, T is the full NN t-matrix (provided one assumes the absence of anti-nucleons in the deuteron) and the nature of the πNN vertex is explained below. Kowalski et al. pointed out that the inclusion of the crossed term (depicted in Fig. 2) in $t^{(1)}$ leads to double-counting, as follows. In a time-dependent perturbation theory the contribution of this part of $t^{(1)}$ to the diagram on the right of Fig. 1 is the diagram shown in Fig. 3. However, this diagram has already been included as distortion in the initial channel, via the diagram on the left of Fig. 1. Note that in a time-ordered perturbation theory this double-counting problem does *not* arise, since the only contribution made by Fig. 2 to the right-hand diagram of Fig. 1 is the diagram shown in Fig. 4. In a time-ordered approach this diagram is not included as distortion in the incoming NN channel and so is not over-counted.

The existence of this problem raises two questions. Firstly, why does this erroneous double-counting occur in a method which Taylor claimed worked *regardless of the perturbation scheme* being used? Secondly, how can the over-counting be eliminated? Avishai and Mizutani attempted to provide answers to both of these questions in their 1983 paper [10]. They claimed that the double-counting problem occurred because the derivation of the $NN - \pi NN$ equations had only considered the s-channel structure of the amplitudes in question. (We are using the notation of Mandelstam here [16,17].) They suggested that examining the s-, t- and u-channel structure simultaneously would remove the double-

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[1]Note that Avishai and Mizutani included both a term to account for heavy-meson exchange and a three-body force in their calculation, whereas Afnan and Blankleider included neither of these effects. However, the heavy-meson exchanges and a three-body force can easily be included in Afnan and Blankleider's derivation, and when that is done the resulting equations are exactly those obtained by Avishai and Mizutani

counting. However, this proposal contradicts Taylor's original work in which he derived the classification-of-diagrams technique so as to have no double-counting *whichever channel or channels the amplitudes' structure was examined in.* He only proposed performing the structure examination in a number of channels simultaneously as part of an approximation he intended to use in order to close the set of equations obtained from his method. We shall see below that Avishai and Mizutani's suggestion is partly right: the lack of specification of the t- and u-channel cut-structure of the sub-amplitudes from which the full amplitude is constructed can be thought of as the cause of the double-counting problem. Consequently, if the double-counting is to be eliminated the cut-structure of these amplitudes in channels other than the s-channel does need to be considered. But, to cut in all channels simultaneously is unnecessary. Furthermore, such a solution is impractical, as it results in highly non-linear equations.

Another "solution" suggested by Avishai and Mizutani is just to ignore the double-counting, since (they claim) "...compared with the important role played by the direct nucleon pole term in nuclear π absorption, the possible over counting of the crossed pole term should hardly affect the essential physics." [10]. The accuracy of this statement is open to question and the validity of such an approach was never tested, since the numerical calculation based on Avishai and Mizutani's original work used Blankenbecler-Sugar reduction [18] in order to reduce the four-dimensional equations, thus time-ordering them and eliminating the double-counting difficulty [19]. Therefore this "solution" is practical, but, since it amounts to ignoring the problem completely we question whether it really is a solution at all!

The example presented by Kowalski et al. [15] shows that double-counting can occur when the Taylor method is applied to a time-dependent perturbation theory of the $NN-\pi NN$ system. The presence of this double-counting indicates a fundamental flaw in the Taylor method, which must be resolved before the method can be used confidently in order to derive four-dimensional equations for *any* system. In this paper we solve this double-counting problem in general, by first pointing out why the double-counting arises in Taylor's method, and then demonstrating how correction terms can be introduced to eliminate it. Consequently, we answer both the questions which were posed about the double-counting problem above.

Therefore, this paper resolves two issues associated with Taylor's method: the validity of TRAB's simplification of Taylor's original work and the double-counting problem. In order to do this we first recapitulate Taylor's original argument, in Section II. In so doing we find that certain instances of double-counting arise if Taylor's method is not applied carefully. However, we show that this type of double-counting may be eliminated if we constrain the cut-structure of amplitudes in channels other than the s-channel. Then, in Section III we compare Taylor's result for an arbitrary amplitude with the result obtained from TRAB's conceptually simpler approach. In Section IV two examples of double-counting, including the example originally provided by Kowalski et al. [15], are given. The flaw in Taylor's method which leads to these two instances of double-counting is then discussed, as is the way in which this type of double-counting differs from that discussed in Section II. Finally, in Section V a general solution to this second type of double-counting problem is provided.

II. THE CLASSIFICATION-OF-DIAGRAMS METHOD OF TAYLOR: A REVIEW

In order to place the two points made in this paper about the classification-of-diagrams technique in their proper context we first review the Taylor method, summarizing the arguments presented in Taylor's original paper [1]. In this review we examine the method as applied in the s-channel, but with appropriate modifications the technique may be applied in any channel.

Taylor's method is a topological procedure which allows the summation of a series of diagrams via the classification of these diagrams according to their irreducibility. The method does not assume that these diagrams have been generated by a field theory. The diagrams could, for example, be diagrams representing the perturbation series expansion for an interacting system of N particles. However, in this paper we take the view that the diagrams under consideration are Feynman diagrams generated by some field theory. In this view, the Taylor method provides a means of deriving equations connecting the amplitudes obtained from this underlying field theory.

Therefore, we assume there exists some perturbation expansion of Feynman diagrams, which when summed give a set of $m \to n$ Green's functions, which in momentum-space we represent by:

$$G_{n\leftarrow m}(p_1', p_2', \ldots, p_n'; p_1, \ldots, p_m). \tag{1}$$

If, of the $m(n)$ particles in the initial (final) state, j (j') are nucleons and the rest are pions, LSZ reduction [20] may be used to obtain the amplitude corresponding to this Green's function:

$$\begin{aligned} A^{(j',j)}_{n\leftarrow m}(p_1', \ldots, p_n'; p_1, \ldots, p_m) d_N^{-1}(p_1') \ldots d_N^{-1}(p_{j'}') d_\pi^{-1}(p_{j'+1}') \ldots d_\pi^{-1}(p_n') \\ \times\, G^{(j',j)}_{n\leftarrow m}(p_1', \ldots, p_n'; p_1, \ldots, p_m)\ d_N^{-1}(p_1) \ldots d_N^{-1}(p_j) d_\pi^{-1}(p_{j+1}) \ldots d_\pi^{-1}(p_m)\ . \end{aligned} \tag{2}$$

(Note that the use of the terms "initial" and "final" state here, and in the ensuing argument, is slightly liberal, since in time-dependent perturbation theory there is nothing which restricts the times associated with the m-particles with reference to those associated with the n-particles. But, by "initial" state we mean the state with m particles having momenta $p_1, \ldots, p_m$ and by "final" state we mean the state with n particles, having momenta $p_1', \ldots, p_n'$.) Taylor's method provides a way of classifying all the perturbation diagrams contributing to $A_{n\leftarrow m}$ according to their topology.

However, Taylor's method works only if all the particles involved are fully dressed. In order, therefore, for us to be able to discuss the Taylor method, we need to assume that all particles are fully dressed. (For a discussion of how this renormalization might be achieved see [21].)

Furthermore, in order to simplify matters as much as possible, we consider only distinguishable particles. Equations for indistinguishable particles may then either be obtained by symmetrizing or anti-symmetrizing the equations for distinguishable particles in the usual way, or by making the necessary changes to the Taylor method in order for it to apply to indistinguishable particles. Taylor himself pursued the latter approach in his original work [1]. For examples of the former approach see Ref. [3] or the papers [13,22].

The classification-of-diagrams technique is then based on the following definitions, which apply to any perturbation diagram, regardless of the perturbation scheme used to construct

the diagram. (Note that the definitions would have to be suitably modified if we intended to consider the structure of the amplitude in any channel other than the s-channel.)

Definition (r-cut) *An r-cut is an arc which separates initial from final states and intersects exactly r lines, at least one of which must be an internal line. If all of the r lines cut are internal lines then the cut is called an internal r-cut.*

Note that in writing this definition we assume that all perturbation diagrams are represented in a two-dimensional plane. We do allow the lines in any diagram to "jump over" one another: two lines do not have to meet at an interaction vertex whenever they intersect. By contrast, a cut is defined to intersect all the lines it meets: it may not jump over any of them. (Other definitions of an r-cut, which do not assume that the diagrams are represented in the plane, may be composed but it is the above definition which Taylor himself used.)

Definition (r-particle irreducibility) *A diagram is called r-particle irreducible if, for all integers $0 \leq k \leq r$, no k-cut may be made on it. An amplitude is called r-particle irreducible if all diagrams contributing to it are r-particle irreducible.*

Using these two definitions any diagram contributing to the connected $(r-1)$-particle irreducible $m \rightarrow n$ amplitude, $A^{(r-1)(c)}_{n \leftarrow m}$, may be placed in one of five classes. The class of the diagram is determined by what r-cuts may be made on it. The r-cuts which may be made on this particular diagram are first divided as follows: if an r-cut is not internal it is called "initial" if it intersects at least one initial-state but no final-state line; "final" if it intersects at least one final-state, but no initial-state line, and "mixed" if it intersects both initial and final-state lines. The criteria for placing the diagram in one of the classes C_1–C_5 may now be stated as follows:

C_1: No r-cut may be made on the diagram, i.e. it is r-particle irreducible;

C_2: At least one internal r-cut may be made on the diagram, but no mixed or final r-cut is possible;

C_3: Only initial r-cuts are possible;

C_4: At least one mixed r-cut may be made, but no final r-cut is possible;

C_5: At least one final r-cut may be made.

The process of choosing which class to place a diagram in is represented by the flowchart in Fig. 5. This flow chart makes it clear that any perturbation diagram must belong to one and only one class. Therefore, we may sum each of C_1 to C_5 separately, and then express $A^{(r-1)(c)}_{n \leftarrow m}$ as the sum of the five expressions we thereby obtain.

Now, while class C_1 may be summed directly, the classes C_2–C_5 must each be summed by exhibiting a unique latest r-cut in each diagram and so splitting the diagram into an r-particle irreducible part and an $(r-1)$-particle irreducible part. This is done via the following lemma, known as the Last Internal Cut Lemma (LICL).

Lemma (Last Internal Cut) *Any $(r-1)$-particle irreducible diagram which admits an internal r-cut has a unique internal r-cut which is nearest to the final state.*

We now rehearse Taylor's proof of this result, since the structure of the proof will turn out to be important in understanding the double-counting problem. The proof is based on that given by Taylor [1], but has been slightly modified in order to (we hope!) make its structure clearer.

Proof: Consider any two internal r-cuts c_1 and c_2. We wish to find an internal r-cut as late or later than both of them. If the cuts do not intersect it is clear which of the two is earlier and which later, and the later of the two cuts is thus the internal r-cut we are looking for. If they do intersect we define c_1^- and c_2^- to be the portions of the two cuts nearest the initial state and c_1^+ and c_2^+ to be the portions of the two cuts nearest the final state. Even if the cuts intersect each other more than once we may still proceed in this way. If an odd number of intersections occurs then all but the first and last intersection are ignored; c_1^+ and c_2^+ (c_1^- and c_2^-) are defined to be the portions of c_1 and c_2 which are later (earlier) than the last (first) intersection. If an even number of intersections occur the portions of the two cuts between any two intersections are compared individually and the resulting sequence of pieces of cut joined to form c_1^+, c_2^+, c_1^- and c_2^-[2]. We then construct $c^- = c_1^- \cup c_2^-$ and $c^+ = c_1^+ \cup c_2^+$[3]. These definitions of c^- and c^+ do not, however, tell us in which of the two sets to place a line that is cut by *both* c_1 and c_2. In order to resolve this ambiguity we proceed as follows. The diagram under consideration may be distorted so that any line which is cut by both c_1 and c_2 is either intersected by both cuts while it is horizontal, or intersected by both cuts when it is vertical. Lines which fall into the first category are called horizontal in $c_1 \cap c_2$, and lines which fall into the second category are called vertical in $c_1 \cap c_2$. The sets c^- and c^+ are then defined to both contain any line which is horizontal in $c_1 \cap c_2$, and to both not contain any line which is vertical in $c_1 \cap c_2$. Fig. 6 provides a pictorial example of these definitions. Note that it is necessary to define c^+ and c^- in this way in order that they are completely separated from one another, with c^+ nearer to the final state than c^-.

Now, clearly c^+ is nearer to the final state than either c_1 or c_2. It is also clearly an internal cut, since it is composed entirely of internal lines. But, is it an r-cut? Denote by $N(c)$ the number of lines cut by an arc c. Then:

$$N(c_1) = r; \quad N(c_2) = r \; . \tag{3}$$

Furthermore, since the diagram in question is $(r-1)$-particle irreducible and c^+ and c^- both constitute cuts on it, we have:

$$N(c^-) \geq r; \quad N(c^+) \geq r \; . \tag{4}$$

But, because of the way c^- and c^+ are defined:

[2] Since diagrams are two-dimensional objects these definitions are unambiguous.

[3] Note that the use of set notation here corresponds to viewing these cuts as sets whose members are the lines they cut.

$$c_1 \cup c_2 \supseteq c^- \cup c^+ \ ; \tag{5}$$

and

$$c_1 \cap c_2 \supseteq c^- \cap c^+ \ . \tag{6}$$

Now, using:

$$N(A \cup B) = N(A) + N(B) - N(A \cap B) \tag{7}$$

in Eq. (5), and applying Eq. (3) gives:

$$2r - N(c_1 \cap c_2) \geq N(c^-) + N(c^+) - N(c^- \cap c^+) \ . \tag{8}$$

But Eq. (6) implies that:

$$N(c_1 \cap c_2) \geq N(c^- \cap c^+) \ . \tag{9}$$

The only way Eqs. (8), (9) and (4) can be reconciled is if:

$$N(c^-) = N(c^+) = r \ . \tag{10}$$

Thus c^- and c^+ are both internal r-cuts, and so we have achieved our aim of finding an internal r-cut as late or later than both c_1 and c_2. Note that (10) taken with (8) and (9) implies that:

$$N(c_1 \cap c_2) = N(c^- \cap c^+) \ . \tag{11}$$

When combined with (6) this gives:

$$c_1 \cap c_2 = c^- \cap c^+ \ . \tag{12}$$

Since c^- and c^+ do not contain any line which is vertical in $c_1 \cap c_2$, it follows that no line which is cut by c_1 and c_2 can be vertical in $c_1 \cap c_2$.

Applying the above procedure many times allows the construction of a unique latest r-cut, which is nearest to the final state. Note that this last cut lemma apples only to *internal* r-cuts on $(r-1)$-*particle irreducible* diagrams, which is why we must be careful to distinguish between class C_2, in which the latest r-cut will obviously be an internal r-cut, and classes C_3, C_4 and C_5, in which an r-cut that cuts at least one external line may be the latest r-cut.

Given the above argument, it is clear that we could easily also prove the following result:

Lemma (First Internal Cut) *Any $(r-1)$-particle irreducible diagram which admits an internal r-cut has a unique internal r-cut which is nearest to the initial state.*

Armed with these two lemmas we now proceed to sum the five classes C_1–C_5.

A. Class C_1

The sum of all r-particle irreducible diagrams contributing to $A^{(r-1)(c)}_{n\leftarrow m}$ is clearly $A^{(r)(c)}_{n\leftarrow m}$, the connected r-particle irreducible $m \to n$ amplitude. This, therefore, is the sum of class C_1.

B. Class C_2

The latest r-cut in any diagram in C_2 must be an internal r-cut. Therefore, it is apparent that by using the last internal cut lemma we may express any connected $(r-1)$-particle irreducible diagram, $a^{(r-1)}_{n\leftarrow m}$, which belongs to C_2 as:

$$\left[a^{(r)}_{n\leftarrow r} \; G^{(r)} \; a^{(r-1)}_{r\leftarrow m} \right]^{(c)} \quad , \tag{13}$$

where $G^{(r)}$ is the free propagator for r fully-dressed particles, and $a^{(r-1)}_{n\leftarrow r}$ and $a^{(r)}_{r\leftarrow m}$ are two diagrams which are $(r-1)$ and r-particle irreducible respectively. Note that $a^{(r-1)}_{n\leftarrow r}$ and $a^{(r)}_{r\leftarrow m}$ need not be connected, provided that they obey the conditions discussed below. Summing over all diagrams which contribute to $A^{(r-1)(c)}_{n\leftarrow m}$ and are in C_2 then involves summing over all $(r-1)$-particle irreducible, $m \to n$ Feynman diagrams with the structure given in Eq. (13). Consequently, the sum of C_2 is:

$$C_2 = \left[A^{(r)}_{n\leftarrow r} \; G^{(r)} \; A^{(r-1)}_{r\leftarrow m} \right]^{(c)} \quad . \tag{14}$$

See Fig. 7 for a pictorial representation of this sum of class C_2. The amplitudes $A^{(r)}_{n\leftarrow r}$ and $A^{(r-1)}_{r\leftarrow m}$ may both contain disconnected pieces, as long as:

1. The overall amplitude they form is connected (that is what the superscript $^{(c)}$ indicates).

2. The disconnected pieces do not contain any diagrams in which a particle (or particles) propagates freely without interacting with any other particles.

3. The disconnected pieces of $A^{(r)}_{n\leftarrow r}$ and $A^{(r-1)}_{r\leftarrow m}$ are such as do not allow the presence of cuts which cut less than r lines, or r-cuts involving final-state lines. (The presence of such cuts is automatically forbidden if the amplitudes $A^{(r)}_{n\leftarrow r}$ and $A^{(r-1)}_{r\leftarrow m}$ are connected.) See, for example, Fig. 8, which shows a case in which a diagram apparently belonging to the sum (14) admits an $(r-1)$-cut. This $(r-1)$-cut is possible because the disconnected piece of the amplitude $A^{(r)}_{n\leftarrow r}$ may admit an "$(r-1)$-cut" which involves only final and initial-state lines. Such a "cut" is not precluded by the constraint of r-particle irreducibility, but may still lead to an $(r-1)$-cut when $A^{(r)}_{n\leftarrow r}$ is used as part of a larger diagram. Therefore we must extend the notion of r-particle irreducibility in $A^{(r)(d)}_{n\leftarrow r}$ in order to prevent l-"cuts" with $l \leq r$ which involve only initial and final-state lines. A similar extension of the $(r-1)$-particle irreducibility of $A^{(r-1)(d)}_{r\leftarrow m}$ must also be imposed. Once these revised definitions are made the difficulty of undesirable r-cuts, or cuts cutting less than r lines, no longer arises here.

C. Class C_3

Now consider class C_3. We wish to take any diagram in C_3 and find a unique r-cut nearest to the final state. However, in this case the situation is complicated by the fact that only r-cuts which intersect at least one line from the initial state are possible. It is therefore

necessary to eliminate the external lines from consideration before applying the last internal cut lemma.

Consider any diagram contributing to $A^{(r-1)(c)}_{n\leftarrow m}$ and in C_3. Construct the set of all r-cuts which may be made upon the diagram. For each r-cut, define r_i to be the number of lines from the initial state which that cut intersects. Then, take the minimum of r_i over all possible r-cuts on a given diagram and denote the result by t_i. This t_i is then the minimum number of lines from the initial state cut by any r-cut possible in this particular diagram. We call any r-cut which satisfies:

$$r_i = t_i \ , \tag{15}$$

a minimal r-cut, and we denote the set of all minimal r-cuts by M_{t_i}. It is clear that if we can construct a unique latest r-cut, X, out of this set M_{t_i} then no cut in any set M_{r_i} will be later than this cut X. In fact, the following stronger result applies:

Claim: *If $(r_i + t_i) < m$ or $r \leq m$ then if the latest cut in M_{t_i} exists it must be later than all cuts in M_{r_i}.*

Proof: Assume a latest cut $X \in M_{t_i}$ exists. Suppose $Y \in M_{r_i}$ is not earlier than X. Then Y cannot be later than X, therefore X and Y must intersect. So, construct c^+_{XY} and c^-_{XY} using the procedure discussed in the proof of the LICL above. That procedure may, however, break down here since c^-_{XY} need not obey $N(c^-_{XY}) \geq r$ as if X and Y are initial r-cuts, c^-_{XY} may consist entirely of initial-state lines. However, if c^-_{XY} is to consist only of initial-state lines then it must cut all of the m initial-state lines. But, c^-_{XY} can cut at most $(r_i + t_i)$ initial-state lines. Therefore, if:

$$(r_i + t_i) < m \ , \tag{16}$$

c^-_{XY} will definitely not consist only of lines from the initial-state. Furthermore, if $r \leq m$ then the equation $N(c^-_{XY}) \geq r$ is automatically satisfied, even if c^-_{XY} is not actually a cut — again because if c^-_{XY} is not to be a cut then it must intersect *all* initial-state lines. (Actually $r < m$ leads to a contradiction, since the LICL argument shows that if $N(c^-_{XY}) \geq r$ then $N(c^-_{XY}) = r$. therefore a c^-_{XY} containing only initial-state lines is impossible for $r < m$.)

Thus if $r \leq m$ or $(r_i + t_i) < m$ the constructed r-cut c^+_{XY} is later than both X and Y. But, since c^+_{XY} was formed from the cuts X and Y it must cut at most t_i initial-state lines. Thus, either X is not the latest cut in M_{t_i} or t_i is not the minimum number of initial lines cut by an r-cut on the diagram. Either possibility contradicts our assumptions. It follows that $Y \in M_{r_i}$ must be earlier than X.

It is obviously still necessary to construct a latest cut in M_{t_i}. However, to construct such a cut is difficult because the cuts in the set M_{t_i} must be divided as follows. Although all cuts in M_{t_i} must cut the same *number* of lines from the initial state they do not necessarily cut the same *set* of initial lines. Different r-cuts within M_{t_i} may cut different sets of external lines $\tilde{t}_i$, as long as each such set $\tilde{t}_i$ has t_i members. Therefore, the minimal r-cuts must themselves be divided into subsets according to which group of initial-state lines $\tilde{t}_i$ they cut. To this end we construct subsets of M_{t_i}, $M_{\tilde{t}_i}$, with each r-cut in the subset $M_{\tilde{t}_i}$ intersecting a specific set of lines from the initial state, $\tilde{t}_i$. Note that each subset $M_{\tilde{t}_i}$ still may contain many r-cuts. However, within any such subset $M_{\tilde{t}_i}$ there is always a unique latest r-cut,

constructed as follows. Consider any r-cut in $M_{\tilde{t}_i}$, and suppose the lines it intersects form a set s. Now remove the lines $\tilde{t}_i$ from each set s in $M_{\tilde{t}_i}$. This turns all the cuts in $M_{\tilde{t}_i}$ into *internal* $(r-t_i)$ cuts in what may now be regarded as an $(r-t_i-1)$-particle irreducible diagram. Consequently, by the last internal cut lemma, there exists a unique last internal cut, which cuts $(r-t_i)$ lines. By joining the set of lines $\tilde{t}_i$ to this cut we obtain a unique latest r-cut out of all the cuts in this particular subset $M_{\tilde{t}_i}$.

However, in principle there are many sets $\tilde{t}_i$ and so many different "latest" r-cuts will be obtained when the above procedure is applied to the various subsets $M_{\tilde{t}_i}$. It is not immediately clear whether it is possible to construct an *overall* latest minimal r-cut from all these different "latest" minimal r-cuts. Again, one might think that two latest minimal cuts from two different subsets of M_{t_i} could be taken and a cut later than either of them constructed using the procedure outlined in the proof of the last internal cut lemma. As above though, the problem is that c^- may consist entirely of lines from the initial state, and so may not be a true cut at all. However, repeating the proof given above but with r_i set equal to t_i it is clear that if $r \leq m$ or $2t_i < m$ it will definitely be possible to construct a unique latest cut out of the set M_{t_i}. Since $t_i \leq r_i \leq (r-1)$ it follows that if:

$$2(r-1) < m \quad \text{or} \quad r \leq m\ , \tag{17}$$

is satisfied then a unique latest r-cut in the set M_{t_i} exists and is later than all the other cuts possible on the diagram. Since m and r are always positive integers it follows that if:

$$r \leq m \tag{18}$$

then a unique latest r-cut can be found in each C_3 diagram contributing to $A^{(r)}_{n\leftarrow m}$. We now have two possibilities:

1. $r \leq m$, in which case each diagram may be split, in a unique fashion, into an r-particle irreducible and an $(r-1)$-particle irreducible part, and so a sum for C_3 may be constructed.

2. $r > m$, in which case we must be more careful, since a unique latest cut cannot be constructed directly.

Case 1 ($r \leq m$): Suppose that $r \leq m$ and consider any diagram $a^{(r-1)}_{n\leftarrow m}$ which belongs in C_3. For this diagram we may construct the set of minimal r-cuts M_{t_i}, each of which cuts precisely t_i external lines. The above argument then guarantees the existence of a unique last r-cut among all those cuts in M_{t_i}, which is also the overall unique latest r-cut. This cut will cut a certain set of initial lines $\tilde{t}_i$. Applying the procedure described above, of first removing the lines $\tilde{t}_i$ from consideration and then applying the last internal cut lemma, we find the diagram $a^{(r-1)}_{n\leftarrow m}$ may be expressed uniquely as:

$$\left[a^{(r)}_{n\leftarrow r}\ G^{(r/\tilde{t}_i)}\ a^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)}\ , \tag{19}$$

where $G^{(r/\tilde{t}_i)}$ is the free propagator for r fully-dressed particles, but with the particles in the set $\tilde{t}_i$ removed, and the amplitudes $a^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)}$ and $a^{(r)}_{n\leftarrow r}$ may be disconnected, as long as these disconnected pieces are, respectively, r-particle irreducible and $(r-t_i-1)$-particle irreducible in the extended sense discussed above.

We now sum over all diagrams in C_3 contributing to $A^{(r-1)(c)}_{n\leftarrow m}$, but, for the present, restrict the sum to those diagrams for which the minimum number of lines from the initial state which are cut by any possible r-cut is t_i. Since this procedure involves a sum over all topologically distinct diagrams of the form (19), it follows that the result is:

$$C_3^{t_i} = \sum_{\text{All sets } \tilde{t}_i} \left[A^{(r)}_{n\leftarrow r} \; G^{(r/\tilde{t}_i)} \; A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} , \tag{20}$$

where, as above, the amplitudes $A^{(r)}_{n\leftarrow r}$ and $A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)}$ may contain disconnected pieces, provided that:

1. These pieces only represent processes in which each particle interacts with at least one other particle.

2. The irreducibility of the disconnected pieces of these amplitudes is understood in the extended sense discussed above.

3. The overall amplitude formed in expression (20) is connected.

We have denoted the sum of this sub-class of C_3, which includes all diagrams in which the minimum number of lines from the initial state cut by any possible r-cut is t_i, by $C_3^{t_i}$. Clearly then, the sum in Eq. (20) must be restricted to those sets $\tilde{t}_i$ with t_i members. Note also that terms in the sum containing any one-to-one amplitude must always be eliminated, since, due to all particles being fully dressed, we set all one-to-one amplitudes to zero.

Before continuing we observe that, for every diagram in $C_3^{t_i}$, the cut α shown in Fig. 9 must be at least an $(r+1)$-cut, since if it is an r-cut the diagram belongs in C_4, and if it cuts less than r lines then the diagram is not $(r-1)$-particle irreducible, and so cannot contribute to $A^{(r-1)}_{n\leftarrow m}$. It follows that the amplitude $A^{(r)}_{n\leftarrow r}$ in Eq. (20) must be $(r-m+t_i-1)$-particle irreducible in the channels:

$$[n/\tilde{h}] + [r/\tilde{t}_i] \leftarrow [\tilde{h}] + [\tilde{t}_i] \; ,$$

where $\tilde{h}$ is any single-member set of final-state lines. It turns out that this condition is sufficient to stop cuts such as α in Fig. 9 cutting r or less lines. If the condition is not imposed then certain diagrams will be included in both C_3 and C_4 and so will be double-counted. Taylor does not seem to have realized that if this inter-class double-counting is to be avoided, a constraint must be placed on the structure of the amplitude $A^{(r)}_{n\leftarrow r}$ in a channel other than the s-channel. The necessity of this constraint is, in fact, merely a consequence of the fact that in time-dependent perturbation theory, the cut-structure of the factor amplitudes (such as $A^{(r)}_{n\leftarrow r}$) in channels other than the s-channel contributes to the overall s-channel cut-structure of the diagram. Observe that if $r \leq m$ then this constraint is automatically satisfied, due to the s-channel r-particle irreducibility of $A^{(r)}_{n\leftarrow r}$. Hence in the case we are discussing here these constraints are unnecessary. However, they will become necessary if $r > m$.

Another constraint must also be imposed on $A^{(r)}_{n\leftarrow r}$ as follows. If $2t_i > m$, diagrams in which the cut β shown in Fig. 9 is an r-cut should have been included in $C_3^{m-t_i}$ not $C_3^{t_i}$. Therefore, if $2t_i > m$, the condition of $(r-m+t_i)$-particle irreducibility must be imposed on $A^{(r)}_{n\leftarrow r}$ in the channel

$$[n] + [r/\tilde{t}_i] \leftarrow \tilde{t}_i \ . \tag{21}$$

(Note that if this condition is imposed, the condition discussed in the previous paragraph is automatically satisfied.) If this condition is not applied then certain diagrams will be included in both $C_3^{m-t_i}$ and $C_3^{t_i}$ and so inter-sub-class double-counting will occur. Furthermore, even if $2t_i \leq m$ the condition of $(r - m + t_i - 1)$-particle irreducibility must be imposed in this channel, since otherwise diagrams which are not $(r-1)$-particle irreducible are included in the sum of $C_3^{t_i}$. Thus regardless of the relative value of $2t_i$ and m, the amplitude $A^{(r)}_{n\leftarrow r}$ acquires a constraint in the channel (21). Again, this constraint is automatically satisfied if $r \leq m$, due to the s-channel r-particle irreducibility of $A^{(r)}_{n\leftarrow r}$.

Now, diagrams in C_3 contributing to $A^{(r)}_{n\leftarrow m}$ can have any number, t_i, of initial lines cut by the r-cut in question, from a minimum of $t_i = 1$ up to a maximum of $t_i = (r-1)$. Therefore, it follows that if condition (17) is satisfied:

$$C_3 = \sum_{t_i=1}^{r-1} C_3^{t_i} \ , \tag{22}$$

with $C_3^{t_i}$ being given by Eq.(20).

Case 2 ($r > m$): Our earlier discussion made it clear that in this case the existence of a unique latest r-cut could not be guaranteed. However, even if $r > m$ the sub-class $C_3^{t_i}$ may still be summed, provided that $2t_i < m$, as follows. If $2t_i < m$ then a unique latest r-cut in the set M_{t_i} may be found in each diagram in $C_3^{t_i}$. This cut may not be the unique latest r-cut in the diagram but it is uniquely defined, and no later r-cut is possible. Consequently this r-cut provides an unambiguous way of splitting each diagram in $C_3^{t_i}$ into two halves. When we sum over all diagrams in $C_3^{t_i}$ we obtain the result:

$$C_3^{t_i} = \sum_{\text{All sets } \tilde{t}_i} \left[A^{(r)}_{n\leftarrow r\tilde{t}_i} \ G^{(r/\tilde{t}_i)} \ A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} \ , \tag{23}$$

where we have had to impose on $A^{(r)}_{n\leftarrow r}$ the restrictions discussed above, since they are no longer automatically satisfied. Note that the sum here is only over those minimal sets $\tilde{t}_i$ with t_i members.

Now, if $2t_i \geq m$ and $r > m$ then we must find a different way to sum $C_3^{t_i}$, since no unique latest r-cut in M_{t_i} exists. Taylor claims that in this case $C_3^{t_i}$ may be summed by splitting it into sub-sub-classes $C_3^{\tilde{t}_i}$, where $C_3^{\tilde{t}_i}$ is defined to be the set of all diagrams belonging to class C_3 in which a minimal r-cut cutting the set of initial lines $\tilde{t}_i$ is possible. So, consider any diagram in $C_3^{\tilde{t}_i}$. As we saw above, once a diagram and a minimal set of lines $\tilde{t}_i$ is chosen, there exists a unique latest r-cut in that diagram, which intersects the set of lines $\tilde{t}_i$. If the procedure described above is applied, of first removing the external lines $\tilde{t}_i$ from consideration and then applying the last internal cut lemma, we find this diagram may be written exactly as in Eq. (19). Consequently, when we sum over all diagrams in $C_3^{\tilde{t}_i}$ we obtain:

$$C_3^{\tilde{t}_i} = \left[A^{(r)}_{n\leftarrow r\tilde{t}_i} G^{(r/\tilde{t}_i)} A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} \ , \tag{24}$$

where the subscript $\tilde{t}_i$ again indicates that the conditions discussed above have been imposed on $A^{(r)}_{n\leftarrow r}$, in order to prevent the undesirable cuts which would otherwise be possible.

Taylor claims that by summing over all possible minimal sets $\tilde{t}_i$ one obtains the sum of all diagrams in C_3. We shall see in Section IV that this mistaken claim is precisely the origin of the double-counting problem mentioned in the Introduction. However, if we, for the present, continue on the basis of this assumption, we find that Eq. (23) still holds. We may make the identification of the sum in Eq. (23) with a sum over all possible sets of initial lines with t_i members. This is possible because we are summing over all topologically distinct diagrams and considering distinguishable particles. Consequently, if the contribution from one set of initial lines $\tilde{t}_i$ is included, the contribution from all other possible sets of initial lines with t_i members must also be included. This identification shows that, *given the assumption* $C_3^{t_i} \stackrel{?}{=} \sum C_3^{\tilde{t}_i}$ (where the sum is defined to run over all minimal sets $\tilde{t}_i$ with t_i members), the sum of $C_3^{t_i}$ is exactly the same in the case $r > m$ as in the case $r \leq m$, but with restrictions imposed on $A^{(r)}_{n\leftarrow r}$ in channels other than the s-channel.

If $r > m$, t_i may take on values from 1 to $(m-1)$. Therefore, it follows that, if condition (17) is not satisfied, the sum of class C_3 may be written as:

$$C_3 = \sum_{t_i=1}^{m-1} C_3^{t_i} \ , \tag{25}$$

where the sum of $C_3^{t_i}$ is still given by Eq. (23).

The sum of C_3 in both of the above cases is represented diagrammatically in Fig. 10. Even though we have not been able to find a single unique latest cut in all diagrams in C_3, the argument given above appears to show that we may express C_3 as the sum of a number of terms, in each of which there is a different unique latest cut.

D. Class C_4

The argument used above to construct the sum of class C_3 is very similar to that used to find the sum of C_4. Consider any diagram in C_4, and consider any r-cut which can be made on that diagram. For this r-cut, r_i is defined as above, and r_f is defined to be the number of lines from the final state which the cut intersects. Then, once more, t_i is defined to be the minimum of r_i, with the minimum taken over all possible r-cuts, and s_f is defined to be the maximum of r_f, with the maximum taken over all r-cuts satisfying $r_i = t_i$. This defines a set of r-cuts, known as minimal/maximal r-cuts, $M_{s_f t_i}$ which all obey the condition:

$$r_i = t_i \quad \text{and} \quad r_f = s_f . \tag{26}$$

Claim: *If there is a latest r-cut in the set of minimal/maximal r-cuts then this cut is later than any r-cut cutting $r_i > t_i$ initial-state lines and $r_f < s_f$ final-state lines, provided that:*

$$(r_i + t_i) < m \quad \textit{or} \quad r \leq m \tag{27}$$

and

$$(r_f + s_f) < n \quad \textit{or} \quad 2r < n + \min\{m, r\} \ . \tag{28}$$

Proof: As above, if the latest minimal/maximal r-cut X is not already later than the cut Y, which cuts r_i initial-state and r_f final-state lines, then we attempt to construct a cut c^+_{XY} later than both. The LICL procedure allows us to do this, provided that:

1. c^-_{XY} does not consist solely of lines from the initial state;
2. c^+_{XY} does not consists solely of lines from the final-state, i.e. it is not really a cut at all.

The condition preventing possibility 1 from occurring is:

$$(r_i + t_i) < m \ . \tag{29}$$

As above, possibility 1 may occur without invalidating the condition $N(c^-_{XY}) \geq r$, provided that $r \leq m$.

Clearly, possibility 2 cannot occur if $(r_f + s_f) < n$. Another condition under which possibility 2 is forbidden can be derived, as follows. We begin by replacing the equation $N(c^-_{XY}) \geq r$, used in the proof of the last internal cut lemma, by:

$$N(c^-_{XY}) \geq \begin{cases} m & \text{if } r > m \\ r & \text{if } r \leq m \end{cases} , \tag{30}$$

since c^-_{XY} may now consist entirely of lines from the initial state. Combining this result with Eqs. (8) and (9) then gives:

$$N(c^+_{XY}) \leq 2r - \min\{m, r\} \ . \tag{31}$$

It follows that if c^+_{XY} is going to consist entirely of lines from the final state, and so invalidate the use of the last internal cut lemma argument, the condition:

$$n \leq 2r - \min\{m, r\} \ , \tag{32}$$

must be satisfied. Thus possibility 2 cannot arise if:

$$(r_f + s_f) < n \quad \text{ or } \quad n + \min\{m, r\} > 2r. \tag{33}$$

This proves the claim.

Since $r_f, r_i \leq (r-1)$ and $s_f, t_i \geq 1$, it follows that if the condition:

$$r \leq m \quad \text{ and } \quad r < n \tag{34}$$

is satisfied then the latest r-cut in the set of minimal/maximal r-cuts will be later than any other r-cut possible on the diagram. Note that if $n = r$ then the constructed latest "r-cut" will merely be the set of final-state lines, therefore we require $r < n$, in which case we find that no "cut", c^+_{XY}, consisting entirely of final-state lines can be formed from two r-cuts X and Y.

Once again, we now seek the latest r-cut within the set of minimal/maximal r-cuts. This may be done, using the LICL procedure, provided that the conditions 1 and 2 discussed above are met. As was seen in the previous subsection, condition 1 leads to the requirement:

$$2t_i < m \quad \text{or} \quad r \le m \ . \tag{35}$$

Condition 2 will automatically be satisfied if:

$$2s_f < n \ . \tag{36}$$

Furthermore, as above, condition 2 will also be satisfied, provided that Eq. (32) is obeyed. Consequently the condition under which the second of the two above possibilities becomes forbidden is:

$$2s_f < n \quad \text{or} \quad 2r < n + \min\{m, r\} \ . \tag{37}$$

Since t_i and s_f are both less than or equal to $(r-1)$ it follows that the condition which guarantees that there is a unique latest r-cut for each diagram in C_4 is merely Eq. (34).

Therefore the argument used in the proof of the last internal cut lemma may definitely be used to construct a unique latest cut out of all the minimal/maximal r-cuts, and this cut will also be the unqiue latest r-cut of all cuts in this diagram if Eq. (34) is satisfied. In this case it is guaranteed that each diagram contributing to $A^{(r)}_{n\leftarrow m}$ will have a unique latest r-cut. Condition (34) is, in fact, a less stringent condition for the success of the last internal cut lemma argument than the condition which was used by Taylor:

$$2(r-1) < n, m \ . \tag{38}$$

We now consider two cases:

Case 1 ($r \le m$ and $r < n$): The existence of a unique latest r-cut allows, via the use of techniques similar to those used to sum the sub-class $C_3^{t_i}$, the summation of all diagrams in the sub-class $C_4^{s_f t_i}$. The sub-class $C_4^{s_f t_i}$ is defined to contain all diagrams in C_4 in which the minimal/maximal r-cut intersects s_f lines from the final state and t_i lines from the initial state. Its sum is:

$$C_4^{s_f t_i} = \sum_{\mathrm{All\ sets\ \tilde{s}_f\ \&\ \tilde{t}_i}} \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \ G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} \ A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} \ , \tag{39}$$

where the sum is constrained to be only over those sets $\tilde{s}_f$ and $\tilde{t}_i$ containing s_f and t_i members respectively, and all one-to-one amplitudes are to be set to zero. Here, $G^{(r/(\tilde{s}_f \cup \tilde{t}_i))}$ is the free propagator for r fully-dressed particles, but with any particle which is in either of the two sets $\tilde{t}_i$ and $\tilde{s_f}$ removed. It follows that the sum of C_4 can be constructed by summing over all possible values of s_f and t_i, yielding:

$$C_4 = \sum_{s_f=1}^{r-1} \sum_{t_i=1}^{r-1} C_4^{s_f t_i} \ , \tag{40}$$

where the sum over s_f and t_i is restricted to those s_f and t_i which obey:

$$(s_f + t_i) \le (r-1) \ . \tag{41}$$

Case 2 ($r \ge n$ or $r > m$): On the other hand, if condition (34) is not satisfied, we may sum classes $C_4^{s_f t_i}$ which obey $2s_f < n$ and $2t_i < m$, by exhibiting the unique latest cut

in the set of minimal/maximal r-cuts. Again, this cut is not the unique latest cut on the diagram, but there is no later cut, and the cut is uniquely defined. We find that if $r \geq n$ or $r < m$ but $2s_f < n$ and $2t_i < m$ then $C_4^{s_f t_i}$ is still given by Eq. (39), subject to certain restrictions discussed below.

Furthermore, if $2s_f \geq n$ and $2t_i \geq m$, and $2s_f - n + 2t_i - m > 0$, then the cut indicated in Fig. 11, which may be made on any diagram in $C_4^{s_f t_i}$, is an l-cut, with $l < r$. Therefore it follows that we can never have:

$$(2s_f - n + 2t_i - m) > 0 \ , \tag{42}$$

or otherwise an l-cut with $l < r$ will be possible on every diagram in $C_4^{s_f t_i}$. Consequently if $2s_f \geq n$ and $2t_i \geq m$ the situations:

$$2s_f > n \quad \text{and} \quad 2t_i \geq m \tag{43}$$

or

$$2s_f \geq n \quad \text{and} \quad 2t_i > m \ , \tag{44}$$

are forbidden, and only $(2s_f - n + 2t_i - m) = 0$ is allowed.

If $2s_f \geq n$ or $2t_i \geq m$, while $(2s_f - n + 2t_i - m) \leq 0$ Taylor claims that we may still sum C_4 by splitting it into sub-sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$. Here $C_4^{\tilde{s}_f \tilde{t}_i}$ is defined to be the set of all diagrams belonging to C_4 for which some minimal/maximal r-cut intersects the lines $\tilde{t}_i$ from the initial state and the lines $\tilde{s}_f$ from the final state. Once again, similar arguments to the above allow Taylor to show that when all contributions to $C_4^{\tilde{s}_f \tilde{t}_i}$ are summed:

$$C_4^{\tilde{s}_f \tilde{t}_i} = \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \ G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} \ A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} \ ; \tag{45}$$

from which he obtains:

$$C_4^{s_f t_i} \stackrel{?}{=} \sum_{\text{All sets } \tilde{s}_f \ \& \ \tilde{t}_i} \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \ G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} \ A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right]^{(c)} \ , \tag{46}$$

where the sum is restricted to those sets $\tilde{s}_f$ and $\tilde{t}_i$ which contain, respectively, s_f and t_i members. Again, we note that the following facts about this result:

1. We question Taylor's moving from Eq. (45) to Eq. (46), for the reasons to be detailed in Section IV.

2. All one-to-one amplitudes in the sum must be set to zero.

Note that in both (39) and (46), the factor amplitudes $A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)}$ and $A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)}$ may be disconnected, provided that the disconnected parts are, respectively, $(r - s_f)$ and $(r - t_i - 1)$-particle irreducible in the extended sense discussed above.

Furthermore, in order to stop diagrams which should be in C_5 also being included in C_4, and so being double-counted, certain restrictions must be placed on these sub-amplitudes in channels other than the s-channel. In fact, the presence of the cut α shown in Fig. 12 shows that $A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)}$ must be $(r - n + s_f)$-particle irreducible in the channel:

$$[\tilde{s}_f] \leftarrow [(r-t_i)/\tilde{s}_f] + [m-t_i] \ .$$

Note that this condition is automatically satisfied if $r < n$, due to the s-channel $(r-t_i-1)$-particle irreducibility of $A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)}$.

A similar problem arises because the cut β shown in Fig. 12 may be drawn. If $2t_i > m$, diagrams in which this cut is an r-cut should have been placed in the sub-class $C_4^{1\,(m-t_i)}$. Thus, if $2t_i > m$ we must enforce the restriction of $(r-m+t_i-1)$-particle irreducibility on the amplitude $A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)}$ in all channels:

$$[(r-s_f)/\tilde{t}_i] + [(n-s_f)/\tilde{h}] \leftarrow [\tilde{h}] + [\tilde{t}_i] \ ,$$

where $\tilde{h}$ is any single-member set of final-state lines. Note that even if $2t_i \leq m$ the amplitude must be $(r-m+t_i-2)$-particle irreducible in this channel, as otherwise a cut involving less than r lines will be possible on some diagrams summed in $C_4^{s_f t_i}$. Note also that if $r \leq m$ this condition is satisfied automatically, due to the s-channel $(r-s_f)$-particle irreducibility of $A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)}$. If $r > m$ the presence of this restriction in the expression for $C_4^{s_f t_i}$ is necessary, in order to ensure that we do not include any diagrams in this sub-class which should actually have been included in other sub-classes of $C_4^{s_f t_i}$ (and so produce inter-sub-class double-counting), or any diagrams which are, in fact, not $(r-1)$-particle irreducible.

Thus, instead of the expression (46) we must write:

$$C_4^{s_f t_i} \stackrel{?}{=} \sum_{\mathrm{All\ sets\ } \tilde{s}_f \ \&\ \tilde{t}_i} \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)_{\tilde{t}_i}} \ G^{(r/(\tilde{s}_f \cup \tilde{t}_i))} \ A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)_{\tilde{s}_f}} \right]^{(c)} \ , \tag{47}$$

where the subscripts $\tilde{t}_i$ and $\tilde{s}_f$ indicate that the restrictions discussed above have been imposed. Note that modification to Eq. (39) is not necessary since if $r < n$ and $r \leq m$ the conditions represented by the subscripts $\tilde{t}_i$ and $\tilde{s}_f$ are automatically satisfied.

The possible values of s_f and t_i may then be summed over in order to yield:

$$C_4 = \sum_{t_i=1}^{\min\{m,r\}-1} \ \sum_{s_f=1}^{\min\{n,r\}-1} C_4^{s_f t_i} \ , \tag{48}$$

where, once again, the sums are restricted to $(t_i + s_f) \leq (r-1)$ and $(m - 2t_i + n - 2s_f) \geq 0$. This result in fact encompasses Eq. (40), which applies only to the case $r \leq m$ and $r < n$. For a diagrammatic representation of the sum of class C_4 see Fig. 13.

E. Class C_5

The method for summing classes C_3 and C_4 is very similar to that used in order to sum class C_5. Consider any diagram in class C_5 and consider any particular r-cut which can be made on that diagram. Define r_f to be the number of lines from the final state cut by that r-cut. The maximum of r_f over all possible r-cuts is taken and is defined to be s_f. The set of r-cuts satisfying:

$$r_f = s_f, \tag{49}$$

is defined to be M_{s_f}, the set of maximal r-cuts. Once again, a unique latest cut may be extracted from this set of r-cuts, and shown to be later than any r-cut involving r_f final-state lines, $r_f < s_f$, provided that the argument used in the proof of the last internal cut lemma is applicable. In the previous section we explained two ways in which this argument might break down when applied to a diagram in class C_4. When the argument is applied to a diagram in class C_5 it cannot break down in the first of these two ways, since, in this case, it is certain that c^- does not contain any initial-state lines. Therefore, the only way the last internal cut lemma argument can fail when applied to a diagram in C_5 is if c^+ contains only lines from the final state. Arguing as we did above shows that c^+ cannot contain only lines from the final state if:

$$(r - 1 + s_f) < n \quad \text{or} \quad r < n \; . \tag{50}$$

Since $s_f \geq 1$, the condition for there to definitely be a unique latest cut among all the cuts in the set M_{s_f}, and for that cut to be later than all other r-cuts possible on the diagram, is found to be:

$$r < n \; . \tag{51}$$

We note that condition (51) is slightly different from the condition used by Taylor. He stated that the condition for the generation of a unique last cut in the set M_{s_f} was $r \leq n$. However, the above discussion shows that the argument used in the proof of the last internal cut lemma may well also fail to generate a unique last cut if $n = r$.

Once more, we now consider two cases:

Case 1 ($r < n$): If condition (51) holds then similar arguments to those used above may be employed in order to show that:

$$C_5 = \sum_{s_f=1}^{r-1} C_5^{s_f} \; ; \tag{52}$$

with:

$$C_5^{s_f} = \sum_{\text{All sets } \tilde{s}_f} \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \; G^{(r/\tilde{s}_f)} \; A^{(r-1)}_{r\leftarrow m} \right]^{(c)} \; , \tag{53}$$

where the sum is restricted to those sets $\tilde{s}_f$ with s_f members, and both $A^{(r-1)}_{r\leftarrow m}$ and $A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)}$ may contain disconnected parts, subject to the restrictions discussed above for disconnected parts.

Case 2 ($r \geq n$): If condition (51) is violated, then provided that $2s_f < n$ the sub-class $C_5^{s_f}$ may still be summed to yield (53), again subject to the corrections discuused below. If $2s_f \geq n$ then Taylor splits C_5 into sub-sub-classes, $C_5^{\tilde{s}_f}$, each of which contains all those diagrams which admit a maximal r-cut intersecting the set of final-state lines $\tilde{s}_f$. When $C_5^{\tilde{s}_f}$ is summed its sum is found to be:

$$C_5^{\tilde{s}_f} = \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \; G^{(r/\tilde{s}_f)} \; A^{(r-1)}_{r\leftarrow m} \right]^{(c)} . \tag{54}$$

By summing over all possible sets $\tilde{s}_f$ with s_f members Taylor obtains Eq. (53) for $C_5^{s_f}$, with the same comments which applied to that result still applying here. Note that, as we did

for C_3 and C_4, we question this last step for the reasons described below. However, if this step is accepted, summing over s_f gives:

$$C_5 = \sum_{s_f=1}^{n-1} C_5^{s_f} \ . \tag{55}$$

Once more, the r-cut depicted in Fig. 14 must be prohibited if $2s_f < n$, and the possibility of l-cuts, with $l < r$ must be stopped, regardless of the value of s_f. Thus we impose the restriction of $(r - n + s_f)$-particle irreducibility if $2s_f < n$, and $(r - n + s_f - 1)$-particle irreducibility if $2s_f \geq n$, on $A^{(r-1)}_{r\leftarrow m}$ in the channel:

$$[\tilde{s}_f] \leftarrow [r/\tilde{s}_f] + [m] \ .$$

Note that since the amplitude is automatically $(s_f - 1)$-particle irreducible in this channel these conditions are automatically satisfied if $r < n$. Therefore we adjust the equation for $C_5^{s_f}$ to read:

$$C_5^{s_f} \stackrel{?}{=} \sum_{\mathrm{All\ sets\ \tilde{s}_f}} \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \ G^{(r/\tilde{s}_f)} \ A^{(r-1)}_{r\leftarrow m\ \tilde{s_f}} \right]^{(c)} \ . \tag{56}$$

For a diagrammatic representation of the sum of class C_5, see Fig. 15.

F. Overall result

This achieves our original aim of finding expressions for each of the classes C_1 to C_5. If we now sum the results of our summation of each of the individual classes we find:

$$\begin{aligned}
A^{(r-1)(c)}_{n\leftarrow m} \stackrel{?}{=} & A^{(r)(c)}_{n\leftarrow m} + \left[A^{(r)}_{n\leftarrow r} G^{(r)} A^{(r-1)}_{r\leftarrow m} \right]^{(c)} \\
& + \sum_{t_i=1}^{\min\{m,r\}-1} \ \sum_{\mathrm{All\ sets\ \tilde{t}_i}} \left\{ A^{(r)}_{n\leftarrow r\tilde{t}_i} G^{(r)} \left[G^{\tilde{t}_i\,-1} A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right] \right\}^{(c)} \\
& + \sum_{s_f=1}^{\min\{n,r\}-1} \ \sum_{t_i=1}^{\min\{m,r\}-1} \ \sum_{\mathrm{All\ sets\ \tilde{s}_f\ \&\ \tilde{t}_i}} \left\{ \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)\tilde{t}_i} G^{\tilde{s}_f\,-1} \right] G^{(r)} \right. \\
& \qquad \qquad \left. \times \ \left[G^{\tilde{t}_i\,-1} A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)\tilde{s}_f} \right] \right\}^{(c)} \\
& + \sum_{s_f=1}^{\min\{n,r\}-1} \ \sum_{\mathrm{All\ sets\ \tilde{s}_f}} \left\{ \left[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} G^{\tilde{s}_f\,-1} \right] G^{(r)} A^{(r-1)}_{r\leftarrow m\ \tilde{s}_f} \right\}^{(c)} ,
\end{aligned} \tag{57}$$

where the superscript $^{(c)}$ indicates that only connected diagrams may be formed. Note that the five terms in this equation are each generated by a different Taylor class, with the nth term generated by C_n, where $n = 1 \ldots 5$ and the question mark above the equals sign reminds us of the queries raised about the validity of certain of the above steps in the case $r \geq n$ or $r > m$.

We observe that in Eq. (57) we have expressed $A^{(r-1)(c)}_{n\leftarrow m}$ in terms of amplitudes of equal or greater irreducibility and amplitudes involving fewer particles. We have done this without

making any assumption about the structure of the underlying field theory, other than the fact that the theory has a perturbation expansion in terms of Feynman diagrams. This model-independence is what makes the Taylor method so powerful and useful.

Indeed, as was mentioned above, the Taylor method is valid even if no underlying field theory exists at all. The only prerequisite for an application of the Taylor method is the presence of a diagrammatic expansion. Therefore, the Taylor method may be applied to a system of N particles, in order to derive equations for the N-particle amplitudes in terms of the n-particle amplitudes, where $n < N$. When used in a three-particle system such a procedure results in Faddeev-type equations [23], and when it is used in a four-particle system this procedure will lead to Yakubovskiǐ-type equations [24]. Thus, not only is the Taylor method an extremely valuable model-independent technique within field theory, but it is also applicable to other problems, e.g. those in the theory of N-particle systems.

III. THE RELATIONSHIP OF TAYLOR'S ORIGINAL METHOD TO THE TRAB SIMPLIFICATION

In the last section we reviewed the Taylor method and showed how it allows us to write the amplitude $A^{(r-1)(c)}_{n\leftarrow m}$ in terms of amplitudes of equal or greater irreducibility and amplitudes involving fewer particles. However, since 1979, a simpler version of the Taylor method has also been used. This simplification was first developed for a time-ordered perturbation theory by Thomas and Rinat [11,12]. It was then applied by Afnan and Blankleider to the $NN-\pi NN$ system [13,14], by Afnan and Pearce to the $\pi N-\pi\pi N$ system [4,5], and by Afnan and Araki to the problem of pion photoproduction on the nucleon and deuteron [6,7,8]. In this paper we refer to this simplified method as the TRAB method, and we begin this section by reviewing the method. It has been suggested, most notably by Avishai and Mizutani [10], that, when applied to a *time-dependent perturbation theory*, the TRAB method of classifying diagrams leads to results different from those obtained using Taylor's original method. In order to establish the exact conditions under which the TRAB and Taylor methods are equivalent we examine the expression obtained for $A^{(r-1)(c)}_{n\leftarrow m}$ in the previous section and compare it to that obtained in this section from the TRAB method.

The TRAB method is a simplification of the Taylor method which was explicitly designed only to apply to a time-ordered perturbation theory without anti-nucleons [12]. In the TRAB method the definition of an r-cut and an r-particle irreducible diagram are exactly those given for Taylor's method in the previous section, but with the restriction that, since TRAB deal only with time-ordered diagrams, cuts can only be vertical lines separating the initial and final states. Once cuts are restricted to vertical lines the last-cut lemma is trivial to prove, and many of the restrictions imposed in Taylor's work in order to guarantee the existence of a unique last cut become unnecessary. The last-cut lemma in the TRAB method may therefore be stated as:

Lemma (TRAB last-cut) *There exists a unique latest r-cut in any time-ordered perturbation theory diagram whose irreducibility k is less than r.*

Of course, strictly speaking, this version of the last-cut lemma is only valid in time-ordered perturbation theory, but this new last-cut lemma is much easier to use than the

older, more general, Taylor version. In order to use the TRAB last-cut lemma to find an equation for the connected amplitude, $A^{(r-1)(c)}_{n\leftarrow m}$, we merely observe that all diagrams contributing to this amplitude must be either r-particle irreducible or r-particle reducible. The sum of the diagrams in the first group is clearly the fully-connected r-particle irreducible amplitude $A^{(r)(c)}_{n\leftarrow m}$. The sum of the diagrams in the second group is, by the last-cut lemma:

$$\left[\bar{A}^{(r)}_{n\leftarrow r} \; G^{(r)} \; \bar{A}^{(r-1)}_{r\leftarrow m} \right]^{(c)} , \tag{58}$$

where the amplitude $\bar{A}$ may contain both connected and disconnected pieces, and the superscript $^{(c)}$ indicates that the overall diagram must be connected. Note that the disconnected part of the amplitude $\bar{A}$ *may* contain diagrams in which one or more particles merely propagate freely. Putting the sums of the two groups of diagrams together implies that the TRAB method gives the following equation for $A^{(r-1)(c)}_{n\leftarrow m}$:

$$A^{(r-1)(c)}_{n\leftarrow m} = A^{(r)(c)}_{n\leftarrow m} + \left[\bar{A}^{(r)}_{n\leftarrow r} \; G^{(r)} \; \bar{A}^{(r-1)}_{r\leftarrow m} \right]^{(c)} . \tag{59}$$

That is:

$$A^{(r-1)(c)}_{n\leftarrow m} = A^{(r)(c)}_{n\leftarrow m} + \left\{ \left[A^{(r)(c)}_{n\leftarrow r} + \bar{A}^{(r)(d)}_{n\leftarrow r} \right] G^{(r)} \left[A^{(r-1)(c)}_{r\leftarrow m} + \bar{A}^{(r-1)(d)}_{r\leftarrow m} \right] \right\}^{(c)} . \tag{60}$$

As mentioned above, the disconnected amplitudes $\bar{A}^{(d)}$ may contain terms in which one or more particles merely propagate freely while the others interact.

This technique was applied by Afnan and Blankleider to the $NN-\pi NN$ and $BB-\pi BB$ problems in what was, apparently, a covariant approach [13,14]. This would appear to be incorrect since the TRAB approach was originally designed to be applied only to a time-ordered perturbation theory without anti-nucleons. The remarkable thing is that Afnan and Blankleider's application of the TRAB technique produced exactly the same equations for the $NN - \pi NN$ system as those obtained by Avishai and Mizutani using the full Taylor method [9,10]. Avishai and Mizutani suggested that this coincidence of equations required investigation. Here we discuss this coincidence and find that it occurs only because Avishai and Mizutani ignored the restrictions on amplitudes in channels other than the s-channel — restrictions which, in the previous section, we found were necessary if the correct equation for $A^{(r-1)}_{n\leftarrow m}$ was to be derived.

In order to establish the connection we rewrite Eq. (60) using the definitions:

$$\bar{A}^{(r-1)(d)}_{r\leftarrow m} \bar{A}^{(r-1)(\tilde{d})}_{r\leftarrow m} + \sum_{t_i=1}^{\min\{m,r\}-1} \sum_{\text{All sets } \tilde{t}_i} G^{\tilde{t}_i\,-1} \left[\bar{A}^{(r-t_i-1)(c)}_{(r-t_i)\leftarrow(m-t_i)} + \bar{A}^{(r-t_i-1)(\tilde{d})}_{(r-t_i)\leftarrow(m-t_i)} \right] \tag{61}$$

and:

$$\bar{A}^{(r)(d)}_{n\leftarrow r} = \bar{A}^{(r)(\tilde{d})}_{n\leftarrow r} + \sum_{s_f=1}^{\min\{n,r\}-1} \sum_{\text{All sets } \tilde{s}_f} \left[\bar{A}^{(r-s_f)(c)}_{(n-s_f)\leftarrow(r-s_f)} + \bar{A}^{(r-s_f)(\tilde{d})}_{(n-s_f)\leftarrow(r-s_f)} \right] G^{\tilde{s}_f\,-1} , \tag{62}$$

where in each case $\bar{A}^{(I)(\tilde{d})}_{f\leftarrow i}$ is the disconnected piece of the I-particle irreducible $i \rightarrow f$ amplitude, but with no particles propagating freely, i.e. every particle interacting with at least one other particle. Having made these definitions it is obvious that:

$$\bar{A}^{(r-t_i-1)(c)}_{(r-t_i)\leftarrow(m-t_i)} + \bar{A}^{(r-t_i-1)(\tilde{d})}_{(r-t_i)\leftarrow(m-t_i)} = A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \; ; \quad t_i = 0, 1, 2, \ldots, \min\{m, r\} - 1 \tag{63}$$

$$\bar{A}^{(r-s_f)(c)}_{(n-s_f)\leftarrow(r-s_f)} + \bar{A}^{(r-s_f)(\tilde{d})}_{(n-s_f)\leftarrow(r-s_f)} = A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \; ; \quad s_f = 0, 1, 2, \ldots, \min\{n, r\} - 1, \tag{64}$$

provided that we interpret the disconnected amplitude $\bar{A}^{(I)(\tilde{d})}_{f\leftarrow i}$ to be I-particle irreducible in the extended sense introduced in the previous section. It follows that Eq. (60) may be rewritten as:

$$A^{(r-1)(c)}_{n\leftarrow m} = A^{(r)(c)}_{n\leftarrow m} + \left\{ \left[A^{(r)}_{n\leftarrow r} + \sum_{s_f=1}^{\min\{n,r\}-1} \sum_{\text{All sets } \tilde{\mathrm{s}}_\mathrm{f}} A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} G^{\tilde{s}_f - 1} \right] \right.$$
$$\left. G^{(r)} \left[A^{(r-1)}_{r\leftarrow m} + \sum_{t_i=1}^{\min\{m,r\}-1} \sum_{\text{All sets } \tilde{\mathrm{t}}_\mathrm{i}} G^{\tilde{t}_i - 1} A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)} \right] \right\}^{(c)} . \tag{65}$$

This equation is to be compared to the equivalent equation obtained from the full Taylor method, Eq. (57). (Note that in using Eq. (57) we are completely ignoring the problem of double-counting between the sub-sub-classes $C^{\tilde{n}}$ of a particular sub-class C^n of some class C. We shall return to this difficulty in the next section.) But, provided that the restrictions represented by the subscripts $\tilde{s}_f$ and $\tilde{t}_i$ are ignored, Eq. (57) may be simplified in order to obtain Eq. (65).

Thus if Eq. (57) is accepted as correct the TRAB method and Taylor's original method produce the same result, provided that the restrictions which were placed on $A^{(r-t_i-1)}_{r-t_i\leftarrow m-t_i}$ for $t_i = 0, 1, 2, \ldots, \min\{m, r\} - 1$ and $A^{(r-s_f)}_{n-s_f\leftarrow r-s_f}$ for $s_f = 0, 1, 2, \ldots, \min\{n, r\} - 1$ are ignored. These restrictions were imposed in order to prevent the illegitimate l-cuts, $l \leq r$, which are otherwise possible on the diagrams representing the sums of C_3–C_5. They do not arise in the TRAB method, since that method sprung from classifying time-ordered perturbation theory diagrams, and so the amplitudes involved do not have their irreducibility constrained in any channel other than the s-channel. Therefore, when applied in a time-dependent perturbation theory, the TRAB method includes certain diagrams which are actually members of C_4 in C_3 as well, and certain diagrams which are in C_5 in C_4 as well. It also includes some diagrams in two sub-classes C^n of a particular class C. Worse still, the TRAB method's failure to produce constraints in channels other than the s-channel may mean that certain diagrams which are not $(r-1)$-particle irreducible are included in C_3, C_4 and C_5. However, if n, m and r are less than or equal to three it can be shown that this final difficulty does not arise. Furthermore, if $n, m, r \leq 3$ the TRAB method does not lead to diagrams being included in more than one sub-class C^n of the same class. Therefore in the case $n, m, r \leq 3$, the only problem with the TRAB method is that it produces expressions for C_3–C_5 which mean that:

$$C_3 \cap C_4 \neq \phi \; ; \tag{66}$$

$$C_4 \cap C_5 \neq \phi \; . \tag{67}$$

Now in their derivation of the $NN - \pi NN$ equations Avishai and Mizutani *did* ignore this difficulty as, to some extent, did Taylor himself. Consequently, it is not in the least surprising that Avishai and Mizutani's application of the Taylor method produced the same $NN - \pi NN$ equations as the TRAB method. However, if the Taylor method had been applied correctly in Avishai and Mizutani's work, and the conditions $C_3 \cap C_4 = \phi$ and

$C_4 \cap C_5 = \phi$ rigorously enforced, then they would have obtained different equations to those derived via the TRAB method.

Consequently, care must be exercised when applying the TRAB method to the classification-of-diagrams in time-dependent perturbation theory. But, in essence the only problem with the TRAB method is a general one with any method based on Taylor's work: all such methods contain the possibility of double-counting, as mentioned in the Introduction. We have already seen that double-counting between different classes or different sub-classes (as well as other problems associated with the existence of undesirable cuts) may arise if care is not taken in the summation of the classes. We now discuss a different sort of double-counting, in which the sum of a particular sub-class includes certain diagrams twice.

IV. THE DOUBLE-COUNTING PROBLEM IN THE TAYLOR METHOD

Taylor's classification-of-diagrams technique was developed in an effort to sum the diagrammatic expansion obtained for some Green's function or amplitude in a perturbation theory, counting each topologically distinct diagram once and only once. In this section we first give two examples which show that, in a time-dependent perturbation theory, such as covariant perturbation theory, the method fails to achieve this aim. I.e., when the Taylor method is used to sum certain series of time-dependent perturbation theory diagrams some diagrams in the series are counted twice. The flaw in Taylor's argument which leads to this double-counting is then explained, and it is shown how that mistake leads directly to the two examples of double-counting given.

Note that the type of double-counting discussed here is fundamentally different from that mentioned above. Even though Taylor appears not to have realized the full extent of this first type of double-counting, he does appear to have eliminated it in certain cases. Indeed, the inter-class and inter-sub-class double-counting mentioned so far may be eliminated by taking care when summing C_3–C_5, and placing appropriate restrictions on the amplitudes involved. By contrast, the double-counting to be discussed here requires a careful re-examination of Taylor's method in order to pinpoint the precise fault involved.

A. Two examples of double-counting in the Taylor method

1. Double-counting in pion absorption on the deuteron

The first example we examine is the double-counting of certain covariant perturbation theory diagrams in theories of pion absorption on the deuteron, a problem first pointed out by Kowalski et al. [15]. Consider two distinguishable nucleons, which in the initial state are labeled $N1$ and $N2$ and in the final state are labeled $N1'$ and $N2'$, and suppose that also present in the initial state is a pion, which we label simply π. We call the $3 \rightarrow 2$ amplitude for this process F. Consider, in particular, the two-particle irreducible part of F, $F^{(2)}$. In the discussion here we ignore the restrictions which should be placed on the amplitudes in the sums of C_3, C_4 and C_5 (they make no difference to the thrust of the argument). Consequently Taylor's method and the TRAB simplification thereof are equivalent. Both give the following equation:

$$F^{(2)} = F^{(3)} + \left\{ \left[F^{(3)} + \sum_{j=1,2} f^{(2)}(j)\, d_j^{-1} \right] d_1 d_2 d_\pi \left[M^{(2)} + \sum_{i=1,2} t_{\pi N}^{(1)}(i)\, d_i^{-1} + T_{NN}^{(1)}\, d_\pi^{-1} \right] \right\}^{(c)} , \tag{68}$$

where M is the connected three-to-three amplitude, $t_{\pi N}(i)$ is the two-body πN t-matrix with nucleon i as a spectator, T_{NN} is the two-body NN t-matrix, $f(j)$ is the πNN absorption vertex with particle Nj' as a spectator and d_1, d_2 and d_π are the single-particle fully-dressed free propagators for nucleon 1, nucleon 2 and the pion. Note that the irreducibilities of all amplitudes are indicated by the bracketed superscript. Note also the use of spectator notation to show the particles involved in the two-body interaction. Eq. (68) is presented pictorially in Fig. 16, which includes an indication of the Taylor class or sub-class that produced each term.

Consider, for the moment, only the product of the two-body t-matrices and the πNN vertices. These give a contribution:

$$\sum_{i=1,2} f^{(2)}(i)\, d_{\bar{i}}\, T_{NN}^{(1)} + \sum_{i,j=1,2} \overline{\delta}_{ij} f^{(2)}(j)\, d_\pi\, t_{\pi N}^{(1)}(i) \tag{69}$$

to $F^{(2)}$. Here $\bar{i}$ is defined by:

$$\bar{i} = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases} . \tag{70}$$

We shall see in the next example that the one-particle irreducible NN t-matrix contains a term representing one-pion exchange with undressed vertices:

$$f^{(2)*}(2)\, d_\pi\, {f^{(2)*}}^{\dagger}(1) , \tag{71}$$

where $f^{(2)*}$ is two-particle irreducible in all channels. (This term is depicted in Fig. 17.) It can also be shown that the one-particle irreducible πN t-matrix $t_{\pi N}^{(1)}(i)$ contains a crossed term:

$$f^{(2)}(i)\, d_{\bar{i}}\, {f^{\dagger}}^{(2)}(i) . \tag{72}$$

(See Fig. 18.) If these portions of the NN and πN t-matrices are substituted into the expression (69), and the results treated in a time-dependent perturbation theory, then both terms in Eq. (69) contain the diagram in Fig. 19. That diagram is double-counted. So, Taylor's method breaks down in this example: at least one diagram is counted twice.

Note also that in time-ordered perturbation theory there is no double-counting since the terms involving the NN and πN t-matrix contribute to different time-orders (compare Fig. 4, where the vertices are now known to be two-particle irreducible, with Fig. 19). It is only in a time-dependent perturbation theory, such as covariant perturbation theory, that the diagram in Fig. 4 becomes equal to the diagram of Fig. 19, and the double-counting problem arises.

2. Double-counting in the calculation of one-pion exchange

Another example of double-counting is provided by the calculation of the one-pion exchange potential in the $NN \to NN$ amplitude. Again, suppose we have two distinguishable nucleons, which in the initial state are labeled $N1$ and $N2$ and in the final state are labeled $N1'$ and $N2'$. When Taylor's method is applied to the amplitude $T^{(2)}$ it gives the following sum of class C_4. (We do not write the sum of the other classes here as they are irrelevant to the argument.)

$$C_4 = f^{(2)}(1)\, d_\pi\, \tilde{f}^{(1)\dagger}(2) + f^{(2)}(2)\, d_\pi\, \tilde{f}^{(1)\dagger}(1)\ , \tag{73}$$

where, in order to stop diagrams belonging to C_5 being included in C_4 we have defined $\tilde{f}^{(1)\dagger}(2)$ to be two-particle irreducible in the $N'(\pi N)$ channel. (See Fig. 20.)

However, the vertices $f^{(2)}$ are automatically one-particle irreducible in the $N(N'\pi)$-channel, since all particles involved are fully dressed. Therefore, both terms in C_4 represent the same diagram, shown in Fig. 21. Therefore, Taylor's method leads to the double-counting of this diagram in the calculation of OPE.

B. Why does this double-counting occur?

These two examples show clearly that double-counting *does* arise when Taylor's method is applied to a time-dependent perturbation theory. The next question is: *why* does it arise?

We begin to answer this question by noting that, in both the cases discussed above, the double-counting problem occurs in Taylor's class C_4. Therefore we now examine the argument used by Taylor in his attempt to sum C_4.

Observe that if the condition:

$$r > m \quad \text{or} \quad r \geq n \tag{74}$$

holds, as it does for each of the two examples above, then Taylor sums C_4 by constructing the sum of the sub-sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$ and applying:

$$C_4^{s_f t_i} = \sum_{\text{All sets } \tilde{\text{s}}_\text{f}\ \&\ \tilde{\text{t}}_\text{i}} C_4^{\tilde{s}_f \tilde{t}_i}\ . \tag{75}$$

However, what Taylor should be trying to construct is the *union* of all sub-sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$, not the *sum*. That is, the correct formula is:

$$C_4^{s_f t_i} = \bigcup_{\text{All sets } \tilde{\text{s}}_\text{f}\ \&\ \tilde{\text{t}}_\text{i}} C_4^{\tilde{s}_f \tilde{t}_i}\ . \tag{76}$$

The operation of summation used in Eq. (75) is different to that of set union, since a diagram which is a member of two different sub-sub-classes $C_4^{\tilde{s}^1_f \tilde{t}^1_i}$ and $C_4^{\tilde{s}^2_f \tilde{t}^2_i}$ is included twice in such a summation, whereas it is included only once in a set union. Consequently, the identification:

$$\bigcup_{\text{All sets } \tilde{\text{s}}_\text{f}\ \&\ \tilde{\text{t}}_\text{i}} C_4^{\tilde{s}_f \tilde{t}_i} = \sum_{\text{All sets } \tilde{\text{s}}_\text{f}\ \&\ \tilde{\text{t}}_\text{i}} C_4^{\tilde{s}_f \tilde{t}_i} \tag{77}$$

may be made if and only if all the sets $C_4^{\tilde{s}_f \tilde{t}_i}$ are disjoint. This means that if condition (74) holds then Taylor's method will produce the correct result for C_4 if and only if all the sub-sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$ are disjoint. If condition (74) holds and the sub-sub-classes are not disjoint then any diagram which is a member of more than one sub-sub-class will be double-counted. Similar results hold for C_3 and C_5: if the condition (18) is violated then C_3 will not be summed correctly unless the sub-sub-classes $C_3^{\tilde{t}_i}$ are disjoint, and if condition (51) is not satisfied and the sub-sub-classes $C_5^{\tilde{s}_f}$ are not disjoint then C_5 will not be summed correctly.

Suppose then that condition (74) holds. What justification is there for assuming that these sub-sub-classes are disjoint? Very little, since, as was mentioned above in Section II, one diagram may have many different latest minimal/maximal r-cuts which cut different minimal and maximal sets of external lines $\tilde{s}_f$ and $\tilde{t}_i$. Such a diagram will, by the definition of the sub-sub-classes, belong to many of the sets $C_4^{\tilde{s}_f \tilde{t}_i}$. Therefore, the different $C_4^{\tilde{s}_f \tilde{t}_i}$ for a fixed s_f and t_i are not necessarily disjoint and so it is not necessarily true that:

$$C_4^{s_f t_i} = \sum_{\mathrm{All\ sets\ \tilde{s}_f\ \&\ \tilde{t}_i}} C_4^{\tilde{s}_f \tilde{t}_i} \ , \tag{78}$$

where the sum is now restricted to those sets $\tilde{s}_f$ and $\tilde{t}_i$ with s_f and t_i members respectively.

However, it is true that the sub-classes $C_4^{s_f t_i}$ (which are defined to be the set of all C_4 diagrams whose minimal/maximal r-cut intersects s_f final and t_i initial lines) *are* disjoint, and so:

$$C_4 = \sum_{s_f t_i} C_4^{s_f t_i} \ , \tag{79}$$

is the right formula for constructing the sum of class C_4, once the correct sums of the sub-classes $C_4^{s_f t_i}$ are known.

The two double-counting problems above provide perfect examples of this difficulty. Consider the first example. For the diagram which is double-counted in this example we have $s_f = t_i = 1$ (see Fig. 22). But, there are four possible pairs of sets $\tilde{s}_f$ and $\tilde{t}_i$. Two of these four pairs are:

$$\tilde{s}_f = \{N1'\} \ ; \quad \tilde{t}_i = \{N2\} \ , \tag{80}$$

$$\tilde{s}_f = \{N2'\} \ ; \quad \tilde{t}_i = \{\pi\} \ . \tag{81}$$

For each *pair of sets* we may construct a unique latest three-cut, as shown in Fig. 22. However, when we attempt to construct the *overall unique latest three-cut* via the technique used in the last internal cut lemma, we fail because the constructed "latest" cut, c^+, does not constitute a cut at all, since neither of the lines $N1'$ or $N2'$ is an internal line. As explained above, it was in order to circumvent this difficulty in the construction of a unique latest r-cut that Taylor constructed the sum of the individual sub-sub-classes $C_4^{\tilde{s}_f \tilde{t}_i}$ and then summed over all possible sub-sub-classes. However, diagrams such as Fig. 22 belong to more than one sub-sub-class (in this case $C_4^{\{N1'\}\{N2\}}$ and $C_4^{\{N2'\}\{\pi\}}$) and so are counted twice in such a summation over sub-sub-classes. Consequently, in this case, Taylor's method does not accurately sum class C_4.

Similarly in the second example a unique latest three-cut cannot be found (see Fig.23). However, the double-counted diagram belongs to both $C_4^{\{N1'\}\{N2\}}$ and $C_4^{\{N2'\}\{N1\}}$ and so is double-counted when the sum of C_4 is constructed by the methods advocated by Taylor. Again, Fig. 23 shows the impossibility of constructing a unique last three-cut in this situation, but the problem is *not*, as is claimed by Taylor, solved by summing over the sub-sub-classes $C_4^{\tilde{s}_f\tilde{t}_i}$, since that summation merely leads to the double-counting of this diagram, and, indeed, of any other diagram which belongs to more than one sub-sub-class.

V. SOLVING THE DOUBLE-COUNTING PROBLEM IN THE TAYLOR METHOD

Having discovered this problem with the Taylor method we now attempt to solve it. In this section we construct a systematic solution to the type of double-counting discussed in the previous section.

Firstly note that, as discussed above, this type of double-counting does not occur in summing over the sub-classes $C_3^{t_i}$, $C_4^{s_f t_i}$ and $C_5^{s_f}$. Once the correct sums of these sub-classes are obtained the formulae (22) or (25), (48) and (52) or (55) actually give the right result for the sums of C_3, C_4 and C_5, since these sub-classes are, by definition, disjoint. The problem occurs in obtaining the sums of these sub-classes in the first place, since the sub-sub-classes of these sub-classes: $C_3^{\tilde{t}_i}$, $C_4^{\tilde{s}_f\tilde{t}_i}$ and $C_5^{\tilde{s}_f}$ are *not* disjoint. In contrast to the double-counting fixed in Section II this is *intra*-sub-class double-counting.

Now, suppose that C^l is a sub-class of C_3, C_4 or C_5, in which we have a certain minimum and/or maximum number of lines from the initial/final state cut, and that:

$$C^l = \bigcup_{j=1}^{N} C^{\tilde{l}_j} \ , \tag{82}$$

where $\{\tilde{l}_j : j = 1, 2, \ldots, N\}$ is a set of N sets of external lines, all with l members. Then it is clear that the sum of C^l may be found by taking each of the sub-sub-classes in turn and adding them one at a time to a "running sum". However, a particular sub-sub-class $C^{\tilde{l}_j}$ may only be added to this running sum after any diagrams already included in the running sum have been removed from it. Mathematically this procedure is expressed as follows:

$$C^l = \sum_{j=1}^{N} C^{\tilde{l}_j} - \sum_{k<j} C^{\tilde{l}_j} \cap C^{\tilde{l}_k} + \sum_{h<k<j} C^{\tilde{l}_j} \cap C^{\tilde{l}_k} \cap C^{\tilde{l}_h} - \ldots \ . \tag{83}$$

Taylor merely ignores all but the first term in this expression, and that is what leads to the intra-sub-class double-counting discussed in the previous section. Note once again that the difference between this type of double-counting and that discussed in Section II is that above we dealt with double-counting between different classes, or between the different sub-classes C^l (inter-class or inter-sub-class double-counting). Here the double-counting is between the sub-sub-classes $C^{\tilde{l}_j}$ of a particular sub-class C^l (intra-sub-class double-counting).

Before dealing specifically with C_3–C_5 we observe that diagrams in the intersection $C^{\tilde{l}_j} \cap C^{\tilde{l}_k}$ will be those which admit two different r-cuts: one involving the set of external lines $\tilde{l}_j$, and one involving the set of external lines $\tilde{l}_k$. Consequently by examining the result for

$C^{\tilde{l}_j}$ and determining which diagrams in that sum admit an r-cut involving the set of lines $\tilde{l}_k$ we may determine the sum of the diagrams in $C^{\tilde{l}_j} \cap C^{\tilde{l}_k}$. This procedure involves examining the overall s-channel cut-structure of the sum of $C^{\tilde{l}_j}$. In time-ordered perturbation theory this cut-structure is fully determined by the s-channel cut-structure of the sub-amplitudes making up the full amplitude. However, in any time-dependent perturbation theory the s-channel cut-structure depends, not only on the s-channel cut structure of the sub-amplitudes, but also on their cut-structure in other channels. Therefore, in order to eliminate double-counting it is necessary to examine the cut-structure of the sub-amplitudes in a number of different channels.

Similar considerations allow us to (if necessary) construct the intersections of three or more sub-classes. That is to say, the procedure for evaluating such intersections is merely an extension of that for finding $C^{\tilde{l}_j} \cap C^{\tilde{l}_k}$. Consequently, we now apply the ideas of the previous paragraph to each class C_3–C_5 in turn, thus showing how to calculate the first correction term in Eq. (83) for any sub-class of C_3–C_5. We leave it to the interested reader to extend this argument to the calculation of further correction terms in Eq. (83).

A. C_3

In this case we wish to calculate:

$$C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}} \ , \tag{84}$$

where $\tilde{t}_i^{(j)}$ and $\tilde{t}_i^{(k)}$ are two minimal sets of initial-state lines, in order to obtain the corrected sum of $C_3^{t_i}$.

As was explained above, if $2t_i < m$ then even if $r > m$ the sum of any diagram in $C_3^{t_i}$ may still be constructed by finding the unique latest cut in M_{t_i}. Thus unless $2t_i \geq m$ and $r > m$ it is not necessary to pursue the construction of sub-sub-classes $C_3^{\tilde{t}_i}$. Consequently the following discussion only applies to the case $2t_i \geq m$. If $2t_i \geq m$ it follows that:

$$\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)} \neq \phi \ . \tag{85}$$

Now, any diagram in $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}}$ admits two r-cuts X and Y:

$$X = \tilde{t}_i^{(j)} \cup \{r - t_i \text{ internal lines}\} \tag{86}$$

$$Y = \tilde{t}_i^{(k)} \cup \{r - t_i \text{ internal lines}\} \ , \tag{87}$$

which are, respectively, the latest r-cuts involving the sets of lines $\tilde{t}_i^{(j)}$ and $\tilde{t}_i^{(k)}$. The set of initial-state lines (m) may be written as:

$$(m) = (m/\tilde{t}_i^{(j)}) \cup (m/\tilde{t}_i^{(k)}) \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \ , \tag{88}$$

therefore if we define:

$$\tilde{I} = (m/\tilde{t}_i^{(j)}) \cap (m/\tilde{t}_i^{(k)}) \tag{89}$$

we have:

$$(m) = [(m/\tilde{t}_i^{(j)})/\tilde{I}] \cup [(m/\tilde{t}_i^{(k)})/\tilde{I}] \cup \tilde{I} \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \ , \tag{90}$$

where all four sets in this union are disjoint. But:

$$(m) = (m/\tilde{t}_i^{(j)}) \cup \tilde{t}_i^{(j)} \ , \tag{91}$$

and comparing this equation with Eq. (90) gives:

$$\tilde{t}_i^{(j)} = [(m/\tilde{t}_i^{(k)})/\tilde{I}] \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \ . \tag{92}$$

(Similarly for $\tilde{t}_i^{(k)}$.) Note that if we write $N(\tilde{I}) = I$ we have:

$$N(\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) = 2t_i - m + I \tag{93}$$

$$N([(m/\tilde{t}_i^{(j)})/\tilde{I}]) = m - t_i - I \ . \tag{94}$$

Eq. (92) then suggests that:

$$X = [(m/\tilde{t}_i^{(k)})/\tilde{I}] \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \cup \{r - t_i \text{ internal lines}\}, \tag{95}$$

$$Y = [(m/\tilde{t}_i^{(j)})/\tilde{I}] \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \cup \{r - t_i \text{ internal lines}\} \ . \tag{96}$$

Eliminating the set $\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}$ from consideration produces two $(r - 2t_i + m - I)$-cuts, X' and Y', which are to be made on a $(2m - 2t_i - I) \to n$ diagram.

The LICL procedure is now used to construct $c^+_{X'Y'}$ and $c^-_{X'Y'}$. The argument given in Section II C above suggests that $c^+_{X'Y'}$ will be an $(r - 2t_i + m - I)$-cut, provided $c^-_{X'Y'}$ does not contain only initial-state lines. But, the only initial-state lines which could be in $c^-_{X'Y'}$ are those in the set:

$$S = [(m/\tilde{t}_i^{(j)})/\tilde{I}] \cup [(m/\tilde{t}_i^{(k)})/\tilde{I}] \ . \tag{97}$$

Since these two sets are disjoint:

$$N(S) = 2m - 2t_i - 2I < 2m - 2t_i - I, \tag{98}$$

provided $\tilde{I} \neq \phi$. Consequently, if $\tilde{I} \neq \phi$, $c^-_{X'Y'}$ cannot consist solely of initial-state lines, and so constructing:

$$c^+_{XY} = c^+_{X'Y'} \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \tag{99}$$

yields an r-cut which is later than both X and Y. Such an r-cut may only cut initial-state lines cut by both X and Y. It follows that c^+_{XY} cuts only initial-state lines from $\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}$, i.e. it cuts only $(2t_i - m + I)$ initial-state lines. But:

$$2t_i - m + I = t_i - (m - t_i - I) < t_i \ , \tag{100}$$

therefore, in this case any diagrams which may be in $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}}$ actually belong to $C_3^{2t_i - m + I}$, not to $C_3^{t_i}$, which is a contradiction. Thus, it follows that we must have:

$$\tilde{I} = \phi \ , \tag{101}$$

if $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}} \neq 0$. Adapting Eq. (92) therefore shows that if $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}} \neq 0$ we must have:

$$\tilde{t}_i^{(j)} = (m/\tilde{t}_i^{(k)}) \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \; ; \tag{102}$$

$$\tilde{t}_i^{(k)} = (m/\tilde{t}_i^{(j)}) \cup (\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}) \; . \tag{103}$$

Once these two equations are derived we are in a position to examine the diagrammatic result for the sum of $C_3^{\tilde{t}_i^{(j)}}$ and derive, using the LICL, the result for $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}}$ shown in Fig. 24. We write this result algebraically as:

$$C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}} = \left[A^{(r)}_{n \leftarrow (2r-m)_{\tilde{t}_i^{(j)} \tilde{t}_i^{(k)}}} \; G^{(r/\tilde{t}_i^{(j)})} \; A^{(r-t_i-1)}_{(r-t_i) \leftarrow (m-t_i)} \; G^{(r/\tilde{t}_i^{(k)})} \; A^{(r-t_i-1)}_{(r-t_i) \leftarrow (m-t_i)} \right]^{(c)} \; , \tag{104}$$

if $(m/\tilde{t}_i^{(j)}) \cap (m/\tilde{t}_i^{(k)}) = \phi$ and:

$$C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}} = 0 \tag{105}$$

otherwise.

Note the following points about this result:

1. The amplitude $A^{(r)}_{n \leftarrow (2r-m)}$ is a more complicated amplitude than $A^{(r)}_{n \leftarrow r}$, since $r > m$ is a necessary condition for double-counting.

2. Furthermore, the notation $A^{(r)}_{n \leftarrow (2r-m)_{\tilde{t}_i^{(j)} \tilde{t}_i^{(k)}}}$ indicates that $A^{(r)}_{n \leftarrow (2r-m)}$ has had the following constraints placed on it:

   (a) Constraints to stop r-cuts involving any number of final-state lines, or cuts intersecting less than r lines. In fact, the constraint of $(r-1-m+t_i)$-particle irreducibility imposed in the channels:

   $$[n/\tilde{h}] + [r/\tilde{t}_i^{(k)}] \leftarrow [\tilde{h}] + [\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}] + [r/\tilde{t}_i^{(j)}]$$

   and

   $$[n/\tilde{h}] + [r/\tilde{t}_i^{(j)}] \leftarrow [\tilde{h}] + [\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}] + [r/\tilde{t}_i^{(k)}] \; ,$$

   where $\tilde{h}$ is any one final-state line, is enough to preclude the possibility of such cuts.

   (b) $(r-t_i)$-particle irreducibility in the channels:

   $$[\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}] + [r/\tilde{t}_i^{(k)}] + [n] \leftarrow [r/\tilde{t}_i^{(j)}]$$

   and

   $$[\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)}] + [r/\tilde{t}_i^{(j)}] + [n] \leftarrow [r/\tilde{t}_i^{(k)}] \; .$$

In fact, both these constraints may be enforced by requiring that $A^{(r)}_{n\leftarrow(2r-m)}$ be $(r-m+t_i)$-particle irreducible in the channels:

$$[n]+[r/\tilde{t}_i^{(k)}] \leftarrow [\tilde{t}_i^{(j)}\cap \tilde{t}_i^{(k)}]+[r/\tilde{t}_i^{(j)}]$$

and

$$[n]+[r/\tilde{t}_i^{(j)}] \leftarrow [\tilde{t}_i^{(j)}\cap \tilde{t}_i^{(k)}]+[r/\tilde{t}_i^{(k)}]$$

as indicated in Fig. 24.

Given this result for $C_3^{\tilde{t}_i^{(j)}}\cap C_3^{\tilde{t}_i^{(j)}}$, if $r>m$ and $2t_i\geq m$ the first correction to the sum of $C_3^{t_i}$ may be calculated via the formula:

$$C_3^{t_i}=\sum_{j=1}^{p}\left(C_3^{\tilde{t}_i^{(j)}}-\sum_{k=1}^{j-1}C_3^{\tilde{t}_i^{(j)}}\cap C_3^{\tilde{t}_i^{(k)}}+\dots\right), \tag{106}$$

where $p=\begin{pmatrix} m \\ t_i\end{pmatrix}$, the sum in the first term is written in Eq. (23) and $C_3^{\tilde{t}_i^{(j)}}\cap C_3^{\tilde{t}_i^{(k)}}$ is given by Eqs. (104) and (105).

B. C_4

We begin by observing that if $2t_i<m$ and $2s_f<n$ the formula (47) gives the correct result for $C_4^{s_f t_i}$. Furthermore, from our discussion in Section II D above we know that we cannot have $(2t_i-m+n-2s_f)>0$. However, if $2t_i\geq m$ or $2s_f\geq n$ while $(2t_i-m+n-2s_f)\leq 0$ we must use the formula:

$$C_4^{s_f t_i}=\sum_{j=1}^{q}\left[C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}-\sum_{k=1}^{j-1}C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}\cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}+\dots\right]. \tag{107}$$

where $q=\begin{pmatrix} n \\ s_f\end{pmatrix}\times\begin{pmatrix} m \\ t_i\end{pmatrix}$. We now calculate the first correction term here for the three different cases which can occur given these conditions:

Case 1: $2t_i\geq m$ and $2s_f<n$

Once again we define:

$$(m/\tilde{t}_i^{(j)})\cap(m/\tilde{t}_i^{(k)})=\tilde{I}\ . \tag{108}$$

We originally assume that $\tilde{I}\neq\phi$ and construct the two r-cuts:

$$X=\tilde{t}_i^{(j)}\cup\{r-s_f-t_i \text{ internal lines}\}\cup\tilde{s}_f^{(j)} \tag{109}$$

$$Y=\tilde{t}_i^{(k)}\cup\{r-s_f-t_i \text{ internal lines}\}\cup\tilde{s}_f^{(k)}\ , \tag{110}$$

which are, respectively, the two latest r-cuts cutting the sets of external lines $\tilde{t}_i^{(j)} \cup \tilde{s}_f^{(j)}$ and $\tilde{t}_i^{(k)} \cup \tilde{s}_f^{(k)}$. Using a similar argument to that applied above we may then show that an r-cut c_{XY}^{+} may be constructed which is later than both X and Y. As above, this leads to a contradiction. Note that in this case it must be checked that c_{XY}^{+} does not consist entirely of final-state lines. This, however, is guaranteed by the condition $2s_f < n$. Thus,

$$\tilde{I} = \phi \tag{111}$$

is a necessary condition if $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} \neq 0$.

We must also consider whether or not $\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi$. Examination of the diagram representing the sum of $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ shows that the existence of an r-cut involving $\tilde{s}_f^{(k)}$ and $\tilde{t}_i^{(k)}$ with $\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} \neq \phi$ must also imply that a cut cutting less than r lines may be made on the diagram (see Fig. 25). Therefore, if $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} \neq 0$ we must have:

$$\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi \ . \tag{112}$$

Once Eqs. (111) and (112) are established applying the LICL to the sum of $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ implies that:

$$C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} = \Bigg[A^{(r-2s_f)}_{(n-2s_f)\leftarrow(2r-m-2s_f)_{\tilde{t}_i^{(j)}\tilde{t}_i^{(k)}}} \; G^{(r/(\tilde{t}_i^{(j)}\cup\tilde{s}_f^{(j)}))} \; A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)_{\tilde{s}_f^{(j)}}}$$
$$\times\, G^{(r/(\tilde{t}_i^{(k)}\cup\tilde{s}_f^{(k)}))} \; A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)_{\tilde{s}_f^{(k)}}} \Bigg]^{(c)} , \tag{113}$$

if $(m/\tilde{t}_i^{(j)}) \cap (m/\tilde{t}_i^{(k)}) = \phi$ and $\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi$, while:

$$C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} = 0 \ , \tag{114}$$

otherwise. This result is depicted in Fig. 26.

Note the following facts about this result:

1. $A^{(r-t_i-1)}_{(r-t_i)\leftarrow(m-t_i)_{\tilde{s}_f}}$ has the same restrictions on it as in Section II D.

2. $A^{(r-2s_f)}_{(n-2s_f)\leftarrow(2r-m-2s_f)_{\tilde{t}_i^{(j)}\tilde{t}_i^{(k)}}}$ must be $(r-m+t_i-s_f)$-particle irreducible in the two $[r-t_i-s_f]+[n-2s_f] \leftarrow [r-t_i-s_f]+[2t_i-m]$-channels indicated in Fig. 26.

Case 2: $2t_i < m$ and $2s_f \geq n$

Essentially, in this case the roles of initial and final-state lines in the above argument are reversed. We construct:

$$(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) = \tilde{I} \ , \tag{115}$$

and show:

$$\tilde{s}_f^{(k)} = [(n/\tilde{s}_f^{(j)})/\tilde{I}] \cup (\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)}) \ , \tag{116}$$

similarly for $\tilde{s}_f^{(j)}$. Then using the same type of argument employed above we show that the two r-cuts X and Y defined by Eqs. (109) and (110) can only be possible if:

$$\tilde{I} = \phi \tag{117}$$

$$\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)} = \phi \ . \tag{118}$$

It follows that if these two conditions are not satisfied then $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} = 0$. If they are satisfied then the LICL, applied to the sum of $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$, implies that:

$$C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} \Bigg[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)_{\tilde{t}_i^{(k)}}} \ G^{(r/(\tilde{t}_i^{(k)}\cup\tilde{s}_f^{(k)}))} \ A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)_{\tilde{t}_i^{(j)}}} \ G^{(r/(\tilde{t}_i^{(j)}\cup\tilde{s}_f^{(j)}))} \times A^{(r-2t_i-1)}_{(2r-n-2t_i)\leftarrow(m-2t_i)_{\tilde{s}_f^{(j)}\tilde{s}_f^{(k)}}} \Bigg]^{(c)} \ , \tag{119}$$

a result shown diagrammatically in Fig. 27. Here:

1. $A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)_{\tilde{t}_i}}$ has the constraint discussed in Section II D for the case $2t_i < m$ imposed on it.
2. $A^{(r-2t_i-1)}_{2r-n-2t_i\leftarrow m-2t_i\,\tilde{s}_f^{(j)}\tilde{s}_f^{(k)}}$ must be $(r-s_f-t_i-1)$-particle irreducible in the two channels:

   $$[r-t_i-s_f] \leftarrow [r-t_i-s_f] + [m-2t_i] + [2s_f-n]$$

   as is indicated in Fig. 27.

Case 3: $2s_f = n$ and $2t_i = m$

Now consider the case $2s_f = n$ and $2t_i = m$. Once again we may show that if the two r-cuts X and Y, defined by (109) and (110) are both to be possible on a diagram in $C_4^{s_f t_i}$ then we must have:

$$\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi \ , \tag{120}$$

$$\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)} = \phi \ , \tag{121}$$

which, in this case, suggests that:

$$(n/\tilde{s}_f^{(j)}) = \tilde{s}_f^{(k)} \ , \tag{122}$$

$$(m/\tilde{t}_i^{(j)}) = \tilde{t}_i^{(k)} \ . \tag{123}$$

Once this is known it is clear that the r-cut Y is the cut indicated in Fig. 28, which represents the sum of $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$, complete with constraints to stop r-cuts involving final-state lines and

cuts cutting less than r-lines. Therefore, the whole of $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ is double-counted, i.e. if the conditions (120) and (121) are obeyed we have:

$$C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} = C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} = C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}} \ . \tag{124}$$

If the figure representing $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ or $C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}$ is redrawn it becomes clear that the two diagrams for $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ and $C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}$ are, in fact, the same, and so this entire diagram is, indeed, included in both sub-sub-classes. (See Fig. 29.)

C. C_5

In order to calculate the correct sum of C_5 it is necessary to use the equation:

$$C_5^{s_f} = \sum_{j=1}^{\ell} \left[C_5^{\tilde{s}_f^{(j)}} - \sum_{k=1}^{j} C_5^{\tilde{s}_f^{(j)}} \cap C_5^{\tilde{s}_f^{(k)}} + \ldots \right] \ , \tag{125}$$

with $\ell = \begin{pmatrix} n \\ s_f \end{pmatrix}$, whenever $2s_f \geq n$ and $r \geq n$. If $2s_f < n$ or $r < n$ then, as was described above, the whole process of breaking $C_5^{s_f}$ into sub-sub-classes is unnecessary.

The argument for the construction of $C_5^{\tilde{s}_f^{(j)}} \cap C_5^{\tilde{s}_f^{(k)}}$ is again similar to that used previously. Since $2s_f \geq n$ we have:

$$\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} \neq \phi \ , \tag{126}$$

and once more we construct:

$$(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) \equiv \tilde{I} \ , \tag{127}$$

and show that unless $\tilde{I} = \phi$ the two "latest" maximal r-cuts:

$$X = \{r - s_f \text{ internal lines}\} \cup \tilde{s}_f^{(j)} \tag{128}$$

$$Y = \{r - s_f \text{ internal lines}\} \cup \tilde{s}_f^{(k)} \ , \tag{129}$$

cannot both be made on the diagram, as if $\tilde{I} \neq \phi$ and both X and Y are possible then an r-cut involving more than s_f final-state lines is also possible. Therefore,

$$(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) \neq \phi \Rightarrow C_5^{\tilde{s}_f^{(j)}} \cap C_5^{\tilde{s}_f^{(k)}} = 0 \tag{130}$$

but if $(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) = \phi$ then the LICL may be used to show that:

$$C_5^{\tilde{s}_f^{(j)}} \cap C_5^{\tilde{s}_f^{(k)}} = \Bigg[A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \ G^{(r/(\tilde{t}_i^{(k)} \cup \tilde{s}_f^{(k)}))} \ A^{(r-s_f)}_{(n-s_f)\leftarrow(r-s_f)} \\ \times G^{(r/(\tilde{t}_i^{(j)} \cup \tilde{s}_f^{(j)}))} \ A^{(r-1)}_{(2r-n)\leftarrow m_{\tilde{s}_f^{(j)}\tilde{s}_f^{(k)}}} \Bigg]^{(c)} \ , \tag{131}$$

where $A^{(r-1)}_{(2r-n)\leftarrow m\,\tilde{s}^{(j)}_f\,\tilde{s}^{(k)}_f}$ has the restriction of $(r - s_f - 1)$-particle irreducibility in the two channels:

$$[r - s_f] \leftarrow [m] + [r - s_f] + [2s_f - n] \ .$$

(See Fig. 30.)

VI. CONCLUSION

In this paper we have reexamined the Taylor method of classification-of-diagrams. A review of the classification-of-diagrams scheme has been given and two questions regarding this method have been answered.

The first question involved the simplification of the Taylor method developed by Thomas, Rinat, Afnan and Blankleider [12,13]. We found that this method, which was originally derived for use in time-ordered perturbation theory, is, in fact, equivalent to the Taylor method in time-dependent perturbation theory too, provided that when using the full Taylor method we ignore the constraints which should be imposed on the cut-structure of sub-amplitudes in channels other than the s-channel. This explains how Afnan and Blankleider, who used the TRAB method in their work on the $NN - \pi NN$ system [13] still managed to obtain the equations found by Avishai and Mizutani using the "full" Taylor method [10].

Secondly, we showed that the Taylor method double-counts certain diagrams when it is applied in a time-dependent perturbation theory. We found that this double-counting can occur in two ways:

1. **Inter-class or inter-sub-class double-counting:** While reviewing the Taylor method we discovered that certain diagrams are included in more than one class, C, or sub-class, C^n, unless constraints are placed on the amplitudes in channels other than the s-channel. However, if these constraints are imposed this type of double-counting is eliminated. Note that certain diagrams which are not $(r - 1)$-particle irreducible may also be (incorrectly) included in the sum of $A^{(r-1)}_{n\leftarrow m}$ unless such restrictions are applied.

2. **Intra-sub-class double-counting:** Taylor's division of all diagrams within a sub-class C^n into sub-sub-classes $C^{\tilde{n}}$ places some diagrams in more than one sub-sub-class. These diagrams are then double-counted when the sums of all the different sub-sub-classes are added together. We outlined a general procedure by which this type of double-counting can be eliminated. We then used Taylor's own LICL to calculate expressions for part of the sum of the double-counted diagrams.

The double-counting-removal techniques developed in this paper are equivalent to examining the full s-channel cut-structure of the amplitude in question and placing constraints on the cut-structure of the sub-amplitudes contributing to this amplitude in channels other than the s-channel in order to eliminate the double-counting. By contrast, Taylor's original method does not sufficiently constrain the cut-structure of the sub-amplitudes in these other channels—it (almost exclusively) only constrains their s-channel cut-structure. From a topological point of view this under-specification of the cut-structure in channels other

than the s-channel is the reason why Taylor's method leads to double-counting when it is applied in a time-dependent perturbation theory.

The modified Taylor method developed in this paper may now be used in order to derive double-counting-free integral equations for systems of mesons and baryons. In particular, these ideas will be applied to the derivation of equations for the amplitudes in a covariant theory of nucleons and pions in a forthcoming paper [25]. This results in equations for the $NN - \pi NN$ system which are covariant and free from the double-counting problems of previous four-dimensional equations.

ACKNOWLEDGMENTS

We would both like to acknowledge A. N. Kvinikhidze and B. Blankleider, each of whom gave us a hint that something might be wrong with the Taylor method. D. R. P. wishes to acknowledge discussions with S. B. Carr. We are also very grateful to our referee, whose perceptive comments contributed significantly to the content of this paper. We both wish to thank the Australian Research Council for their financial support. D. R. P. holds an Australian Postgraduate Research Award.

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FIGURES

FIG. 1. Two diagrams, both of which give contributions to pion absorption on the deuteron in the Taylor method.

FIG. 2. The crossed term in the πN t-matrix.

FIG. 3. The diagram which Kowalski et al. [15] pointed out was double-counted in certain models of pion absorption on the deuteron.

FIG. 4. The diagram obtained when the crossed term is substituted into the right-hand diagram of Fig. 1 in time-ordered perturbation theory. Note that, in time-ordered perturbation theory, this diagram is *not* included in the left-hand diagram of Fig. 1.

FIG. 5. The classification of some $(r-1)$-particle irreducible $n \leftarrow m$ diagram into one of the classes $C_1 - C_5$ according to the kinds of r-cut which may be made on it.

FIG. 6. Two of the ways in which two cuts, c_1 and c_2, may intersect, and the resulting definitions of $c^+ = c_1^+ \cup c_2^+$ and $c^- = c_1^- \cup c_2^-$ in each of the two cases.

FIG. 7. A diagrammatic representation of the sum of Taylor class C_2.

FIG. 8. An example for the case $n = 3$, $m = r = 4$ in which parts of the expression derived for C_2 are not $(r-1)$-particle irreducible if the disconnected pieces of amplitudes are not carefully defined.

FIG. 9. Two cuts α and β which, if they are r-cuts, will place this diagram, summed in $C_3^{t_i}$, in, respectively, C_4, or (if $2t_i > m$) a different sub-class of C_3. Note that α and β could also cut less than r lines.

FIG. 10. A diagrammatic representation of the sum of Taylor class C_3.

FIG. 11. A diagram in $C_4^{s_f t_i}$, with a cut which cuts less than r lines if $(2t_i - m + 2s_f - n) > 0$.

FIG. 12. Two cuts α and β which, if they are r-cuts, will place this diagram, summed in $C_4^{s_f t_i}$, in, respectively, C_5, or (if $2t_i > m$) a different sub-class of C_4. Note also that these two cuts could cut less than r lines.

FIG. 13. A diagrammatic representation of the sum of Taylor class C_4.

FIG. 14. A cut which, if it is an r-cut, will place this diagram, summed in $C_5^{s_f}$, in a different sub-class of C_5, if $2s_f < n$. Note also that this cut could intersect fewer than r lines.

FIG. 15. A diagrammatic representation of the sum of Taylor class C_5.

FIG. 16. The equation for the two-particle irreducible πNN to NN amplitude, $F^{(2)}$, which is obtained from Taylor's method, with the Taylor classes or sub-sub-classes which produce each term indicated.

FIG. 17. Part of the NN t-matrix.

FIG. 18. The crossed term in the πN t-matrix, with the irreducibility of each vertex indicated.

FIG. 19. One diagram which is double-counted when the Taylor method is used to derive an equation for $F^{(2)}$.

FIG. 20. The two diagrams contributing to C_4 for $T^{(2)}$ in the Taylor method, with the two cuts which will place diagrams in C_5 if they are three-cuts.

FIG. 21. The diagram which both $C_4^{\{N1'\}\{N2\}}$ and $C_4^{\{N2'\}\{N1\}}$ sum to when the Taylor method is applied to $T^{(2)}$.

FIG. 22. The two possible "latest" cuts, c_1 and c_2, which lead to the double-counting of Fig. 19, and the "cuts", c^- and c^+, which are obtained when we attempt to apply the argument used in the proof of the last internal cut lemma in order to construct an overall latest cut.

FIG. 23. The two possible "latest" cuts, c_1 and c_2, which lead to the double-counting of Fig. 21, and the "cuts" which are obtained when we attempt to apply the argument used in the proof of the last internal cut lemma in order to construct an overall latest cut.

FIG. 24. The sum of $C_3^{\tilde{t}_i^{(j)}} \cap C_3^{\tilde{t}_i^{(k)}}$ in diagrammatic form, given that $2t_i \geq m$ and $(m/\tilde{t}_i^{(j)}) \cap (m/\tilde{t}_i^{(k)}) = \phi$.

FIG. 25. If the cut Y is an r-cut then it is clear that the cut Y' will cut less than r lines.

FIG. 26. The sum of the intersection $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}$ in the case $2t_i \geq m$ and $2s_f < n$, given that $(m/\tilde{t}_i^{(j)}) \cap (m/\tilde{t}_i^{(k)}) = \phi$ and $\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi$.

FIG. 27. The sum of the intersection $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}} \cap C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}$ in the case $2t_i < m$ and $2s_f \geq n$, given that $(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) = \phi$ and $\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)} = \phi$.

FIG. 28. The sum of sub-sub-class $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$, and the r-cut X which may be made to obtain this sum, with the r-cut Y which indicates that all diagrams in this sub-sub-class are double-counted in the case $2s_f = n$ and $2t_i = m$.

FIG. 29. The diagram which both $C_4^{\tilde{s}_f^{(j)}\tilde{t}_i^{(j)}}$ and $C_4^{\tilde{s}_f^{(k)}\tilde{t}_i^{(k)}}$ sum to, given that $2s_f = n$, $2t_i = m$, $\tilde{s}_f^{(j)} \cap \tilde{s}_f^{(k)} = \phi$ and $\tilde{t}_i^{(j)} \cap \tilde{t}_i^{(k)} = \phi$.

FIG. 30. The sum of the intersection $C_5^{\tilde{s}_f^{(j)}} \cap C_5^{\tilde{s}_f^{(k)}}$ in diagrammatic form if $2s_f \geq n$ and $(n/\tilde{s}_f^{(j)}) \cap (n/\tilde{s}_f^{(k)}) = \phi$.

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